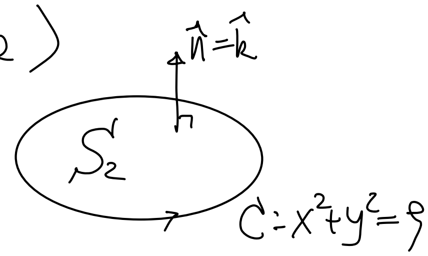
(b) (Same C & same $\vec{F}$, but new surface)

$S_2 : x^2+y^2 \le 9, \quad z=0$

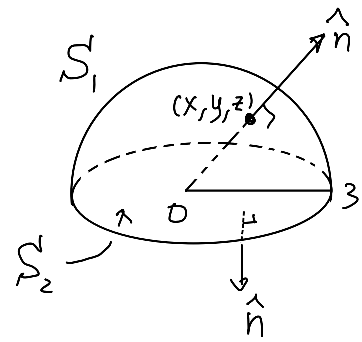


$$\iint_{S_2} (\vec{\nabla}\times\vec{F})\cdot\hat{n}\, d\sigma = \iint_{\{x^2+y^2\le 9\}} (-2\hat{k})\cdot\hat{k}\, dx\,dy$$

$$= -2\,\text{Area}\left(\{x^2+y^2\le 9\}\right) = -18\pi \quad (\text{check!})$$

(c) Same $\vec{F} = y\hat{i} - x\hat{j}$

$S_3 = S_1 \cup S_2$



S_3 has no boundary and in fact encloses a solid region.

Suppose $\hat{n}$ = outward normal of the solid.

$$\iint_{S_3} (\vec{\nabla}\times\vec{F})\cdot\hat{n}\, d\sigma = \iint_{S_1} (\vec{\nabla}\times\vec{F})\cdot\hat{n}\, d\sigma + \iint_{S_2} (\vec{\nabla}\times\vec{F})\cdot\hat{n}\, d\sigma$$

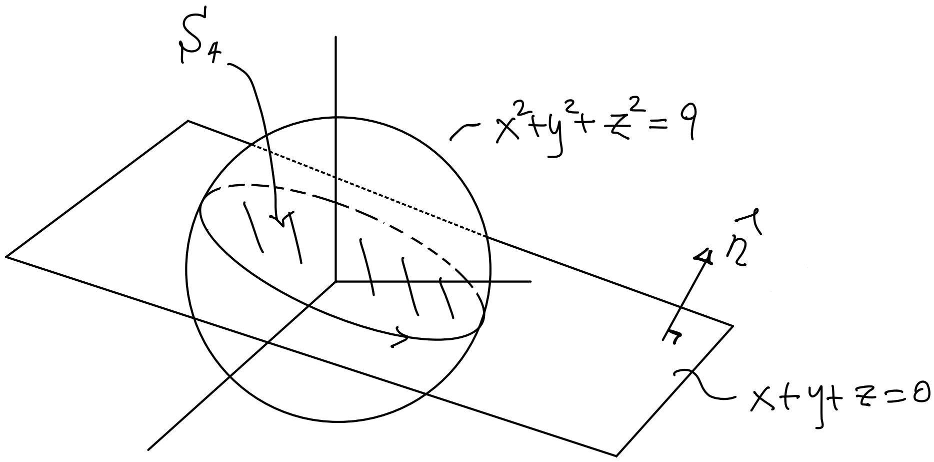
$$= \iint_{S_1} (\vec{\nabla}\times\vec{F})\cdot\hat{n}\, d\sigma + \iint_{S_2} (\vec{\nabla}\times\vec{F})\cdot(-\hat{k})\, d\sigma$$

$$= -18\pi - \iint_{S_2} (\vec{\nabla}\times\vec{F})\cdot\hat{k}\, d\sigma = -18\pi - (-18\pi)$$

$$= 0 \qquad \left(S_3 \text{ has no boundary} \Rightarrow \oint_{\partial S_3} \vec{F}\cdot d\vec{r} = 0\right)$$

#

eg 62 Let $\vec{F} = y\hat{i} - x\hat{j}$ (same $\vec{F}$ as in eg 61, new surface & new boundary curve)



$$S_4 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \leq 9,\ x+y+z=0\}$$

boundary curve of S_4 : $C_4 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2=9,\ x+y+z=0\}$

Find $\oint_{C_4} \vec{F} \cdot d\vec{r}$ (with direction of C_4 given as in the figure.)

Soln: Apply Stokes' Thm

$$\oint_{C_4} \vec{F} \cdot d\vec{r} = \iint_{S_4} (\vec{\nabla} \times \vec{F}) \cdot \hat{n}\, d\sigma$$

$\hat{n}$ is the normal to S_4 is given by the coefficient of the eqt. of the plane

i.e. $\hat{n} = \dfrac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$ +ve $\Downarrow$ upward

$$= \iint_{S_4} (-2\hat{k}) \cdot \frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}\, d\sigma$$

$$= -\frac{2}{\sqrt{3}} \iint_{S_4} d\sigma = -\frac{2}{\sqrt{3}} \text{Area}(S_4)$$

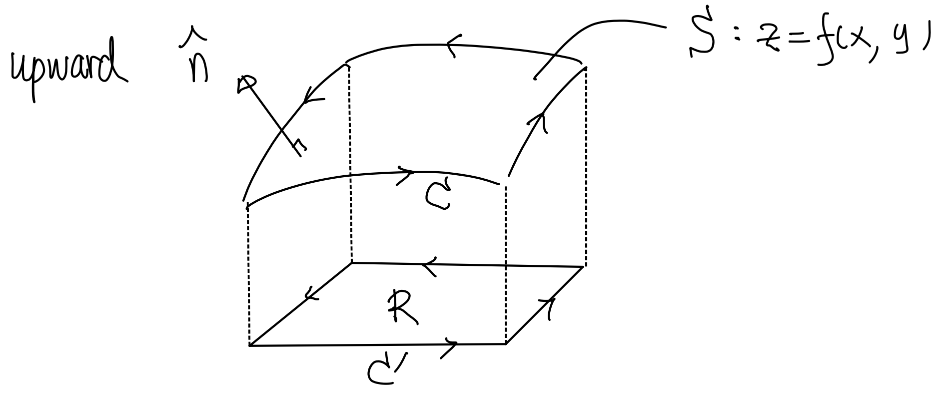
$$= -\frac{2}{\sqrt{3}} (\pi \cdot 3^2) = -\frac{18\pi}{\sqrt{3}}$$

#

Proof of Stokes' Thm

Special case: S is a graph given by

$$z = f(x,y) \quad \text{over a region } R \text{ with } \underline{\text{upward}} \text{ normal}$$



Assume C is the boundary of S, and C' is the boundary of R (anti-clockwisely oriented wrt the normal of S and the plane respectively)

Parametrize the graph as

$$\vec{r}(x,y) = x\hat{i} + y\hat{j} + f(x,y)\hat{k}, \quad (x,y) \in R$$

Then as before

$$\begin{cases} \vec{r}_x = \hat{i} + \dfrac{\partial f}{\partial x}\hat{k} \\ \vec{r}_y = \hat{j} + \dfrac{\partial f}{\partial y}\hat{k} \end{cases}$$

$$\Rightarrow \quad \vec{r}_x \times \vec{r}_y = -\frac{\partial f}{\partial x}\hat{i} - \frac{\partial f}{\partial y}\hat{j} + \hat{k} \quad \text{(upward)}$$

Hence $\quad \hat{n} = \dfrac{\vec{r}_x \times \vec{r}_y}{|\vec{r}_x \times \vec{r}_y|}$ is the upward (unit) normal of S,

and $\quad d\sigma = |\vec{r}_x \times \vec{r}_y|\, dx\, dy = |\vec{r}_x \times \vec{r}_y|\, dA$

area element of R

Let $\vec{F} = M\hat{i} + N\hat{j} + L\hat{k}$ be the C^1 vector field.

Then

$$\iint_S (\vec{\nabla}\times\vec{F})\cdot\hat{n}\, d\sigma = \iint_R (\vec{\nabla}\times\vec{F})(\vec{r}(x,y))\cdot \frac{\vec{r}_x\times\vec{r}_y}{\cancel{|\vec{r}_x\times\vec{r}_y|}}\cdot \cancel{|\vec{r}_x\times\vec{r}_y|}\, dA$$

$$= \iint_R \left[(L_y - N_z)\hat{i} + (M_z - L_x)\hat{j} + (N_x - M_y)\hat{k}\right]\cdot\left[-\frac{\partial f}{\partial x}\hat{i} - \frac{\partial f}{\partial y}\hat{j} + \hat{k}\right] dA$$

$$= \iint_R \left[-f_x(L_y - N_z) - f_y(M_z - L_x) + (N_x - M_y)\right] dx\,dy$$

For the line integral

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C M\,dx + N\,dy + L\,dz$$

parametrized by $(x,y)\in C'$

$$= \oint_{C'} M\,dx + N\,dy + L\,df \qquad (z = f(x,y))$$

$$= \oint_{C'} M\,dx + N\,dy + L(f_x\,dx + f_y\,dy)$$

$$= \oint_{C'} (M + Lf_x)\,dx + (N + Lf_y)\,dy.$$

Precisely: If C' is parametrized by

$$\vec{\gamma}(t) = (x(t), y(t)) \text{ for } a \le t \le b.$$

Then C is parametrized by

$$\vec{r}(t) = (x(t), y(t), f(x(t), y(t)))$$

$$= x(t)\hat{i} + y(t)\hat{j} + f(x(t), y(t))\hat{k}, \quad a \le t \le b$$

$$\Rightarrow \oint_C \vec{F}\cdot d\vec{r} = \int_a^b \begin{bmatrix} M(\vec{r}(t))\,x'(t) + N(\vec{r}(t))\,y'(t) \\ + L(\vec{r}(t))\,\dfrac{d}{dt} f(x(t), y(t)) \end{bmatrix} dt$$

$$= \int_a^b [Mx' + Ny' + L(f_x x' + f_y y')]\, dt$$

$$= \int_a^b [(M + Lf_x)x' + (N + Lf_y)y']\, dt$$

$$= \oint_{C'} (M + Lf_x)\, dx + (N + Lf_y)\, dy$$

Then by Green's Thm

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_{C'} (M + Lf_x)\, dx + (N + Lf_y)\, dy$$

$$= \iint_R \left[\frac{\partial}{\partial x}(N + Lf_y) - \frac{\partial}{\partial y}(M + Lf_x)\right] dA$$

$$= \iint_R \left\{ \begin{array}{l} \frac{\partial}{\partial x}\left[N(x,y,f(x,y)) + L(x,y,f(x,y))\, f_y(x,y)\right] \\ - \frac{\partial}{\partial y}\left[M(x,y,f(x,y)) + L(x,y,f(x,y))\, f_x(x,y)\right] \end{array} \right\} dA$$

$$= \iint_R \left\{ \begin{array}{l} (N_x + N_z f_x) + (L_x + \underline{L_z f_x})\underline{f_y} + \cancel{L f_{yx}} \\ -\left[(M_y + M_z f_y) + (L_y + \underline{L_z f_y})\underline{f_x} + \cancel{L f_{xy}}\right] \end{array} \right\} dA$$

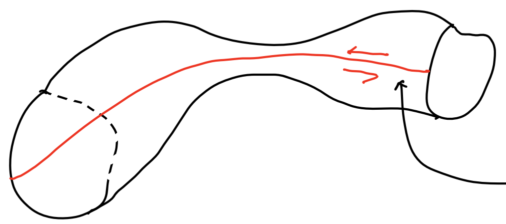
$$= \iint_R \left[-f_x(L_y - N_z) - f_y(M_z - L_x) + (N_x - M_y)\right] dA$$

$$= \iint_S (\vec{\nabla} \times \vec{F}) \cdot \hat{n}\, d\sigma$$

This proves the case of C^2 graph.

<u>General case</u>: Divides S into finitely many pieces which are graphs (in certain projection).

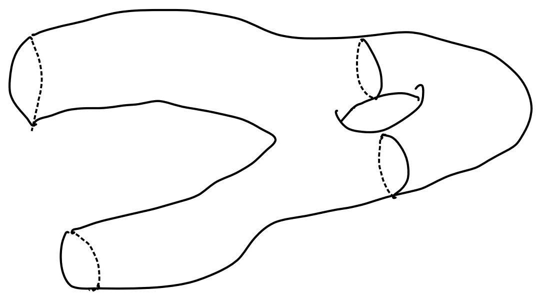
[This includes S with many boundary components as in the Green's Thm



add some curve likes this to make it in 1 bdy component.]

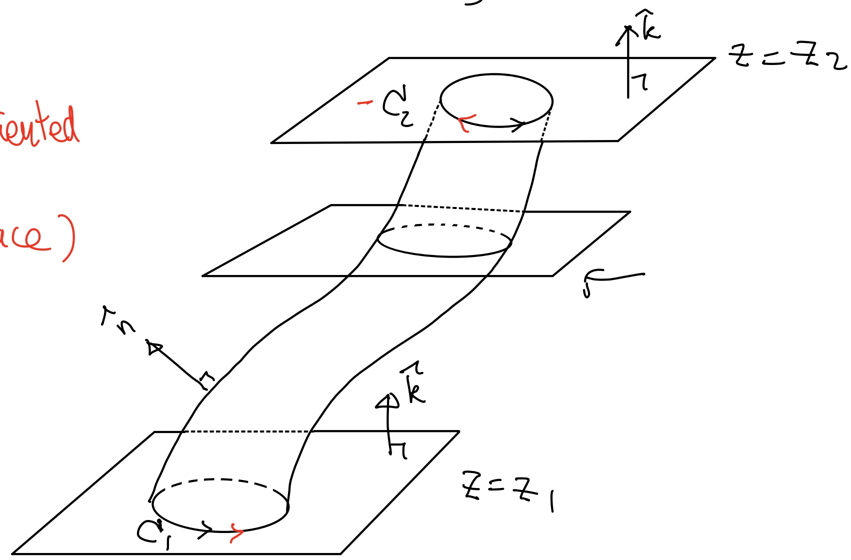
(Proof of general case omitted)

<u>Note</u>: Stokes' Thm applies to surfaces like the following



<u>eg 63</u>: Let $\vec{F}$ be a vector field such that $\underline{\vec{\nabla}\times\vec{F}=0}$ and defined on a region containing the surface S with unit normal vector field $\hat{n}$ as in the figure:

red: oriented wrt $\hat{n}$ (surface)



black: oriented wrt $\hat{k}$ (horizontal plane)

The boundary C of S has 2 components C_1 and C_2 at the level $z=z_1$ & $z=z_2$ respectively.

If both C_1, C_2 oriented anticlockwisely with respect to the "horizontal planes" (i.e. $\hat{k}$).

Then when C oriented with respect to $\hat{n}$ (the surface normal) we have $C = C_1 - C_2$

And Stokes' Thm $\Rightarrow$

$$0 = \iint_S (\vec{\nabla} \times \vec{F}) \cdot \hat{n}\, d\sigma = \oint_C \vec{F} \cdot d\vec{r}$$

oriented wrt $\hat{n}$.

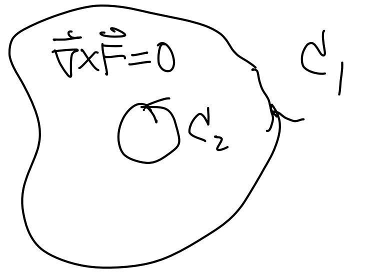
$$= \oint_{C_1} \vec{F} \cdot d\vec{r} - \oint_{C_2} \vec{F} \cdot d\vec{r}$$

oriented wrt "horizontal plane" (i.e. $\hat{k}$)

$$\Rightarrow \oint_{C_1} \vec{F} \cdot d\vec{r} = \oint_{C_2} \vec{F} \cdot d\vec{r}$$

※

Compare this with Green's Thm on plane region with one hole:



$$\Rightarrow \oint_{C_1} \vec{F} \cdot d\vec{r} = \oint_{C_2} \vec{F} \cdot d\vec{r} \qquad \text{(check!)}$$

anti-clockwisely wrt "plane" (not see region as a surface).

Proof of Thm 10 (3-dim'l case)

Only the "$\Leftarrow$" part remains to be proved:

By assumption $\vec{F} = M\hat{i} + N\hat{j} + L\hat{k}$ satisfies the system of eqts in the cor. to the Thm 9, that is

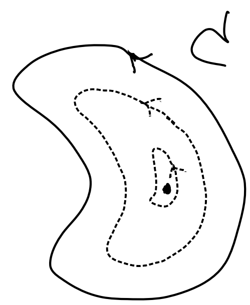
$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \quad \frac{\partial N}{\partial z} = \frac{\partial L}{\partial y}, \quad \text{and} \quad \frac{\partial L}{\partial x} = \frac{\partial M}{\partial z}.$$

Hence $\vec{\nabla} \times \vec{F} = \vec{0}$

Let C be a simple closed curve in a simply-connected region D

Then C can be deformed to a point inside D

The process of deformation gives an oriented surface $S \subset D$ such that the boundary of $S = C$.

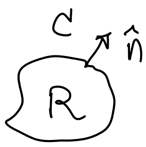
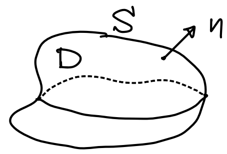


By Stokes' Thm,

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\vec{\nabla} \times \vec{F}) \cdot \hat{n}\, d\sigma = 0 \quad (\text{since } \vec{\nabla} \times \vec{F} = 0)$$

Then Thm 9 $\Rightarrow$ $\vec{F}$ is conservative. #

Summary

$n=2$	$n=3$
Tangential form of Green's Thm $\oint_C \vec{F}\cdot d\vec{r} = \iint_R (\vec{\nabla}\times\vec{F})\cdot\hat{k}\, dA$	**Stokes' Thm** $\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\vec{\nabla}\times\vec{F})\cdot\hat{n}\, d\sigma$
Normal form of Green's Thm $\oint_C \vec{F}\cdot\hat{n}\, ds = \iint_R \vec{\nabla}\cdot\vec{F}\, dA$  $(C=\partial R)$	**Divergence Thm** (next topic) $\iint_S \vec{F}\cdot\hat{n}\, d\sigma = \iiint_D \vec{\nabla}\cdot\vec{F}\, dV$  $(S=\partial D)$
"flux": by definition $\hat{n}$ is the "outward" normal of curve "C" in the "plane".	surface normal of S, "outward" pointing which can be defined when S encloses a solid region D

Thm 13 (Divergence Theorem)

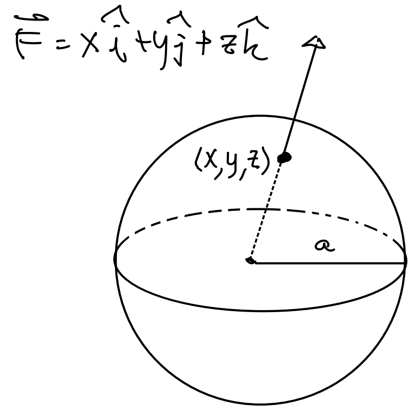
Let $\vec{F}$ be a C^1 vector field on $\Omega \overset{\text{open}}{\subseteq} \mathbb{R}^3$ (no boundary)

S be a piecewise smooth oriented closed surface enclosing a (solid) region $D \subseteq \Omega$.

Let $\hat{n}$ be the outward pointing unit normal vector field on S,

Then $$\boxed{\iint_S \vec{F}\cdot\hat{n}\, d\sigma = \iiint_D \operatorname{div}\vec{F}\, dV = \iiint_D \vec{\nabla}\cdot\vec{F}\, dV}$$

eg 64 Verify Divergence Thm for



$$\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$S = \{x^2+y^2+z^2 = a^2\} \quad (a>0)$$

(surface $= S^2_a$ 2-dim'l sphere of radius a centered at $(0,0,0)$)

$D =$ solid ball bounded by S.

Soln : At $(x,y,z) \in S$

$$\hat{n} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{x^2+y^2+z^2}} = \frac{1}{a}(x\hat{i}+y\hat{j}+z\hat{k})$$ is the outward pointing unit normal

$$\iint_S \vec{F}\cdot\hat{n}\, d\sigma = \iint_S (x\hat{i}+y\hat{j}+z\hat{k}) \cdot \frac{1}{a}(x\hat{i}+y\hat{j}+z\hat{k})\, d\sigma$$

$$= \iint_S a \, d\sigma = a \, \text{Area}(S) = 4\pi a^3 \quad (\text{check!})$$

On the other hand

$$\text{div} \vec{F} = \vec{\nabla} \cdot \vec{F} = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \cdot (x\hat{i} + y\hat{j} + z\hat{k})$$

$$= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$$

$$\Rightarrow \iiint_D \text{div} \vec{F} \, dV = \iiint_D 3 \, dV = 3 \, \text{Vol}(D) = 3 \cdot \frac{4\pi a^3}{3} = 4\pi a^3$$

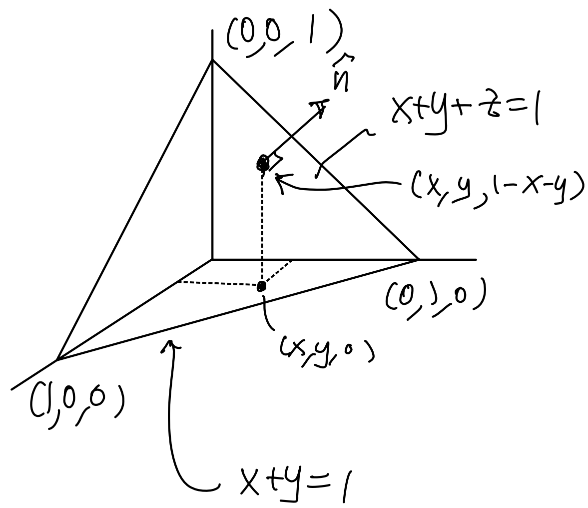
$$= \iint_S \vec{F} \cdot \hat{n} \, d\sigma \qquad \#$$

<u>eg65</u> $\vec{F} = x \sin y \, \hat{i} + (\cos y + z) \hat{j} + z^2 \hat{k}$

Compute outward flux of $\vec{F}$ across the boundary ∂T of

$$T = \left\{ (x,y,z) \in \mathbb{R}^3 : \begin{array}{l} x+y+z \leq 1 \\ x, y, z \geq 0 \end{array} \right\}$$

(tetrahedron)



<u>Soln</u>:

$$\text{div} \vec{F} = \vec{\nabla} \cdot \vec{F} = \frac{\partial}{\partial x}(x \sin y) + \frac{\partial}{\partial y}(\cos y + z) + \frac{\partial}{\partial z}(z^2) = 2z$$

(check!)

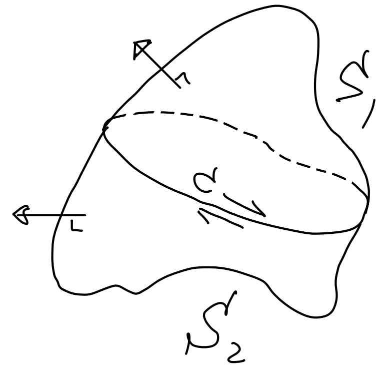
Divergence Thm $\Rightarrow$

$$\iint_{\partial T} \vec{F} \cdot \hat{n} \, d\sigma = \iiint_T \text{div} \vec{F} \, dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} 2z \, dz \, dy \, dx = \frac{1}{12}$$

(check!) $\#$

eg66 : Let S_1, S_2 be 2 surfaces with common boundary curve C such that $S_1 \cup S_2$ forms a closed surface enclosing a solid region D (without hole)

Suppose $\hat{n}$ is the <u>outward</u> unit normal of the (boundary of) solid region D.



Then the <u>orientation</u> of C with respect to $(S_1, \hat{n})$ and $(S_2, \hat{n})$ are <u>opposite</u> (since "$\hat{b}$" of S_1 & S_2 are opposite)

Stokes' Thm $\Rightarrow$

$$\iint_{S_1} (\vec{\nabla}\times\vec{F})\cdot\hat{n}\, d\sigma = \oint_C \vec{F}\cdot d\vec{r} \quad \text{(+ve oriented wrt } (S_1, \hat{n})\text{)}$$

$$= -\oint_C \vec{F}\cdot d\vec{r} \quad \text{(+ve oriented wrt } (S_2, \hat{n})\text{)}$$

$$= -\iint_{S_2} (\vec{\nabla}\times\vec{F})\cdot\hat{n}\, d\sigma$$

$$\Rightarrow \iint_{S_1\cup S_2} (\vec{\nabla}\times\vec{F})\cdot\hat{n}\, d\sigma = 0$$ (see eg61(c) for explicit example)

Divergence Thm $\Rightarrow$

$$\iiint_D \operatorname{div}(\vec{\nabla}\times\vec{F})\, dV = \iint_{S_1\cup S_2} (\vec{\nabla}\times\vec{F})\cdot\hat{n}\, d\sigma = 0.$$

(true for "any" C^2 vector field $\vec{F}$ defined on "any" D)

It is consistent with

(Ex!) $\boxed{\vec{\nabla}\cdot(\vec{\nabla}\times\vec{F}) = 0}$ $\forall$ C^2 vector field

i.e. $\boxed{\operatorname{div}(\operatorname{curl}\vec{F}) = 0}$

Compare: $\boxed{\operatorname{curl}(\operatorname{grad} f) = \vec{0} \quad \text{i.e.} \quad \vec{\nabla}\times(\vec{\nabla}f) = \vec{0}}$

Proof of Divergence Thm

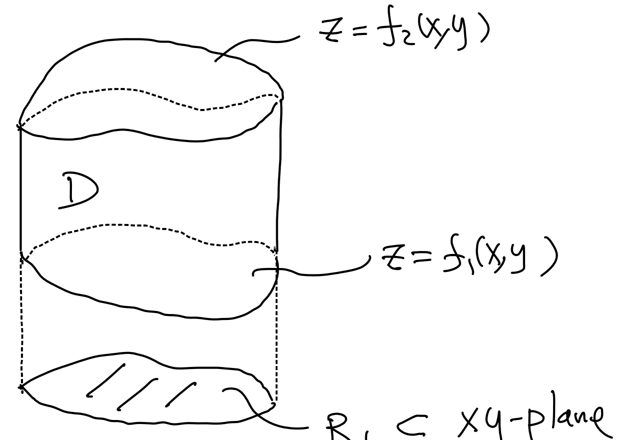
Same as Green's Thm, we'll prove only the case of special domain D which is of type I, II, and III :

$$D = \{(x,y,z) \in \mathbb{R}^3 : (x,y) \in R_1,\ f_1(x,y) \leq z \leq f_2(x,y)\} \quad (\text{type I})$$

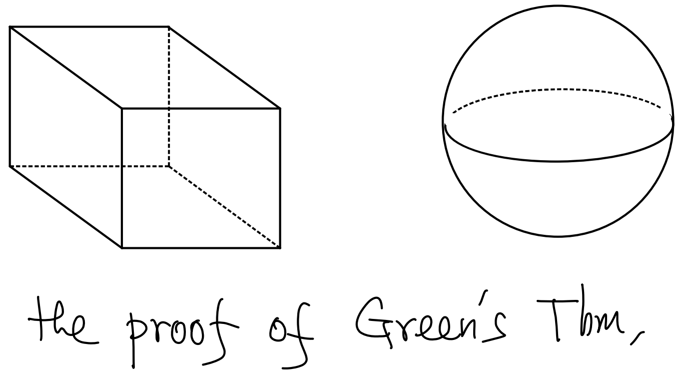
$$= \{(x,y,z) \in \mathbb{R}^3 : (y,z) \in R_2,\ g_1(y,z) \leq x \leq g_2(y,z)\} \quad (\text{type II})$$

$$= \{(x,y,z) \in \mathbb{R}^3 : (x,z) \in R_3,\ h_1(x,z) \leq y \leq h_2(x,z)\} \quad (\text{type III})$$

eg: Type I domain



eg: Special domains:



And also as in the proof of Green's Thm,

for $\vec{F} = M\hat{i} + N\hat{j} + L\hat{k}$

we'll prove 3 equalities in the following which combine to give the divergence thm :

$$\begin{cases} \displaystyle\iint_S M\hat{i}\cdot\hat{n}\,d\sigma = \iiint_D \frac{\partial M}{\partial x}\,dV & \text{(by type II)} \\ \displaystyle\iint_S N\hat{j}\cdot\hat{n}\,d\sigma = \iiint_D \frac{\partial N}{\partial y}\,dV & \text{(by type III)} \\ \displaystyle\iint_S L\hat{k}\cdot\hat{n}\,d\sigma = \iiint_D \frac{\partial L}{\partial z}\,dV & \text{(by type I)} \end{cases}$$

The proofs are similar, we'll prove only the last one:

$$\iint_S L\hat{k}\cdot\hat{n}\,d\sigma = \iiint_D \frac{\partial L}{\partial z}\,dV$$

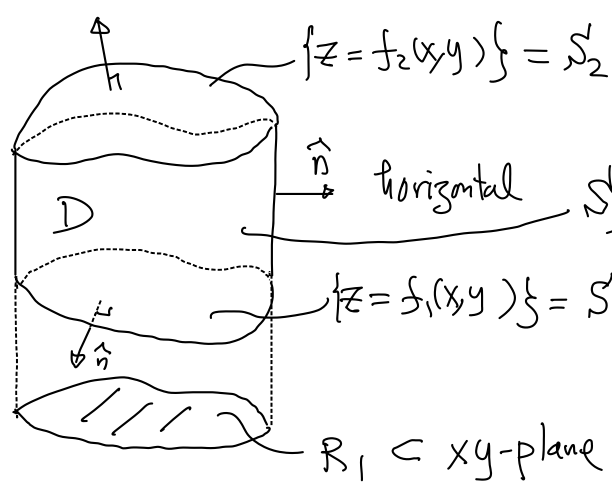
By Fubini's Thm

$$\text{R.H.S.} = \iiint_D \frac{\partial L}{\partial z}\,dV = \iint_{R_1}\left[\int_{f_1(x,y)}^{f_2(x,y)} \frac{\partial L}{\partial z}\,dz\right]dxdy \quad \text{(by type I)}$$

$$= \iint_{R_1}\left[L(x,y,f_2(x,y)) - L(x,y,f_1(x,y))\right]dxdy.$$

For the L.H.S., we note that by definition of type I domain, the boundary surface S of D can be written as

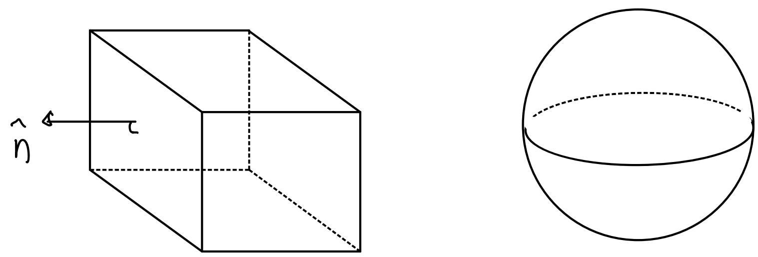
$$S = S_1 \cup S_2 \cup S_3,$$



where $S_1 = \text{graph of } f_1 = \{(x,y,f_1(x,y))\} = \{z = f_1(x,y)\}$

$S_2 = \text{graph of } f_2 = \{(x,y,f_2(x,y))\} = \{z = f_2(x,y)\}$

S_3 = a vertical surface (which could be empty) between S_1 & S_2



S_3 = union of 4 vertical squares

S_3 = empty

Hence

$$\text{L.H.S.} = \iint_S L\hat{k}\cdot\hat{n}\,d\sigma = \iint_{S_1} L\hat{k}\cdot\hat{n}\,d\sigma + \iint_{S_2} L\hat{k}\cdot\hat{n}\,d\sigma + \iint_{S_3} L\cancel{\hat{k}\cdot\hat{n}}\,d\sigma$$

(since $\hat{n}$ of a vertical surface is horizontal, hence $\hat{k}\cdot\hat{n} = 0$)

Now on the upper surface $S_2 = \{z = f_2(x,y)\}$,

the outward normal $\hat{n}$ is upward (in the sense that $\hat{n}\cdot\hat{k} > 0$)

Note that the parametrization

$(x,y) \mapsto \vec{r}(x,y) = x\hat{i} + y\hat{j} + f_2(x,y)\hat{k}$

has $\begin{cases} \vec{r}_x = \hat{i} + \dfrac{\partial f_2}{\partial x}\hat{k} \\ \vec{r}_y = \hat{j} + \dfrac{\partial f_2}{\partial y}\hat{k} \end{cases}$

and $\vec{r}_x \times \vec{r}_y = -\dfrac{\partial f_2}{\partial x}\hat{i} - \dfrac{\partial f_2}{\partial y}\hat{j} + \hat{k}$ +ve $\Rightarrow$ $\vec{r}_x \times \vec{r}_y$ is upward.

Hence $\hat{n} = \dfrac{\vec{r}_x \times \vec{r}_y}{|\vec{r}_x \times \vec{r}_y|}$ is the upward normal

and $\hat{k} \cdot \hat{n} = \dfrac{1}{|\vec{r}_x \times \vec{r}_y|}$.

Therefore $\displaystyle\iint_{S_2} L\hat{k}\cdot\hat{n}\, d\sigma = \iint_{R_1} L(x,y,f_2(x,y)) \cdot \overbrace{\frac{1}{\cancel{|\vec{r}_x \times \vec{r}_y|}}}^{\hat{k}\cdot\hat{n}} \cdot \overbrace{\cancel{|\vec{r}_x \times \vec{r}_y|}\, dA}^{d\sigma}$

$\displaystyle = \iint_{R_1} L(x,y,f_2(x,y))\, dx\, dy$

Similarly, note that the outward normal on S_1 (lower surface) is downward (i.e. $\hat{n}\cdot\hat{k} < 0$), we have (by similar calculation)

$\hat{n} = -\dfrac{\vec{r}_x \times \vec{r}_y}{|\vec{r}_x \times \vec{r}_y|}$, where $\vec{r}(x,y) = x\hat{i} + y\hat{j} + f_1(x,y)\vec{k}$

$\Rightarrow$ $\hat{k}\cdot\hat{n} = -\dfrac{1}{|\vec{r}_x \times \vec{r}_y|}$ (check!)

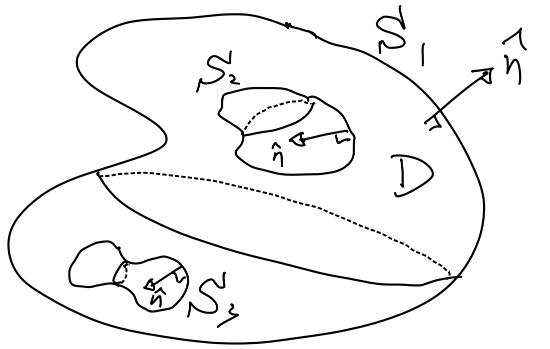
Hence $\displaystyle\iint_{S_1} L\hat{k}\cdot\hat{n}\,d\sigma = -\iint_{R_1} L(x,y,f_1(x,y))\,dxdy$

$$\therefore \iint_{S} L\hat{k}\cdot\hat{n}\,d\sigma = \iint_{R_1}[L(x,y,f_2(x,y)) - L(x,y,f_1(x,y))]\,dxdy$$

$$= \iiint_{D} \frac{\partial L}{\partial z}\,dV.$$

This completes the proof of the divergence thm. ##

<u>Note</u>: Similar to Green's Thm, the Divergence Thm is also hold for solid region with finitely many holes insides:

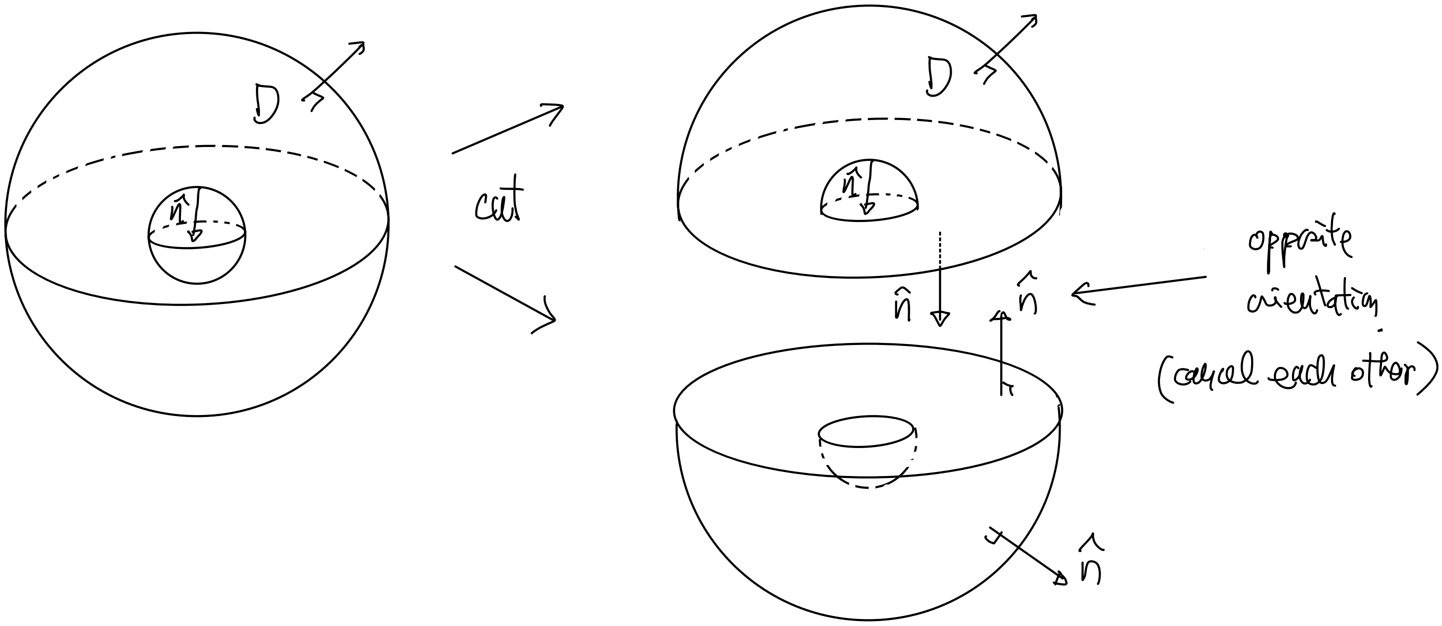


D = solid region inside S_1 but outside of S_2 and S_3

$$\boxed{\iiint_{D} \vec{\nabla}\cdot\vec{F}\,dV = \sum_{i=1}^{n}\iint_{S_i}\vec{F}\cdot\hat{n}\,d\sigma}$$

for $\hat{n}$ = <u>outward</u> normal <u>with respect to D</u>.

eg (Ideal of proof of this kind of surface :)



<u>Note</u>: Physical meaning of $\operatorname{div} F = \vec{\nabla} \cdot \vec{F}$ in $\mathbb{R}^3$

= <u>flux density</u> (by the divergence thm)