Def 19: Let S be *orientable* with *unit normal* $\hat{n}$ (continuous).

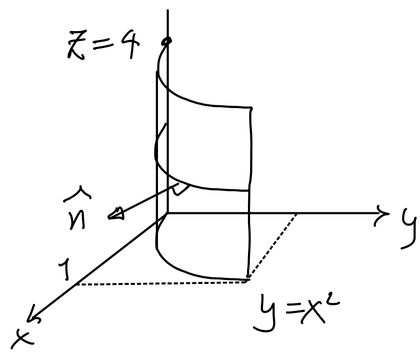
Let $\vec{F}$ be a vector field on S.

Then the *flux* of $\vec{F}$ across S is

$$\text{Flux} = \iint_S \vec{F} \cdot \hat{n} \, d\sigma$$

eg 59: S:

$y = x^2, \ 0 \le x \le 1, \ 0 \le z \le 4$



with $\hat{n}$ given by the natural parametrization

$$\vec{r}(x,z) = x\hat{i} + x^2\hat{j} + z\hat{k}$$

Let $\vec{F} = yz\hat{i} + x\hat{j} - z^2\hat{k}$

Find $\iint_S \vec{F} \cdot \hat{n} \, d\sigma$

Soln: To calculate $\hat{n} = \dfrac{\vec{r}_x \times \vec{r}_z}{|\vec{r}_x \times \vec{r}_z|}$, we have

$$\begin{cases} \vec{r}_x = \hat{i} + 2x\hat{j} \\ \vec{r}_z = \hat{k} \end{cases} \Rightarrow \vec{r}_x \times \vec{r}_z = 2x\hat{i} - \hat{j}$$

$$\therefore \quad \hat{n} = \frac{2x\hat{i} - \hat{j}}{\sqrt{4x^2+1}}$$

Then $$\iint_S \vec{F}\cdot\hat{n}\,d\sigma = \int_0^4\int_0^1 \underbrace{(yz\hat{i}+x\hat{j}-z^2\hat{k})}_{\vec{F}}\cdot \underbrace{\frac{2x\hat{i}-\hat{j}}{\cancel{\sqrt{4x^2+1}}}}_{\hat{n}}\ \underbrace{\cancel{\sqrt{4x^2+1}}\,dx\,dz}_{d\sigma}$$

$$= \int_0^4\int_0^1 (2x^3z - x)\,dx\,dz$$

$$= 2$$

(check!)

#

Remark: $$\iint_S \vec{F}\cdot\hat{n}\,d\sigma = \iint_{(u,v)} \vec{F}(\vec{r}(u,v))\cdot\frac{\vec{r}_u\times\vec{r}_v}{|\vec{r}_u\times\vec{r}_v|}\,|\vec{r}_u\times\vec{r}_v|\,du\,dv$$

$$= \iint_{(u,v)} \vec{F}(\vec{r}(u,v))\cdot(\vec{r}_u\times\vec{r}_v)\,du\,dv$$

Thm 12 (Stokes' Theorem)

Let S be a piecewise smooth oriented surface with piecewise smooth boundary C (including the case that C is a union of finitely many curves). Let

$$\vec{F} = M\hat{i} + N\hat{j} + L\hat{k}$$ be a C^1 vector field.

Suppose C is oriented anti-clockwisely with respect to the unit normal vector field $\hat{n}$ on S. Then

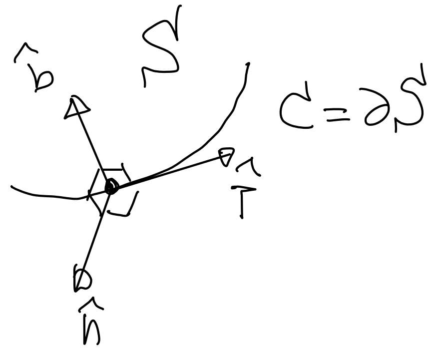
$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F}\cdot\hat{n}\,d\sigma = \iint_S (\vec{\nabla}\times\vec{F})\cdot\hat{n}\,d\sigma$$

Here: (i) if $C = C_1 \cup \cdots \cup C_k$, then it means

$$\sum_{i=1}^{k} \oint_{C_i} \vec{F} \cdot d\vec{r} = \iint_S (\vec{\nabla} \times \vec{F}) \cdot \hat{n} \, d\sigma$$

(ii) "C is oriented anti-clockwisely with respect to the unit normal vector field $\hat{n}$" means that we choose the <u>direction</u> of C such that its (unit) tangent vector $\hat{T}$ satisfies

$\boxed{\hat{b} = \hat{n} \times \hat{T} \text{ pointing } \underline{\text{toward}} \text{ the surface } S}$

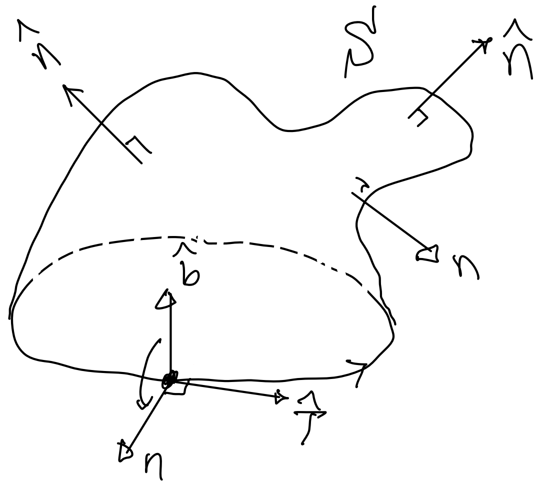


i.e. the unit vector $\hat{b}$ tangent to S, normal to C and pointing toward S satisfies

$$\boxed{\hat{T} = \hat{b} \times \hat{n}}$$

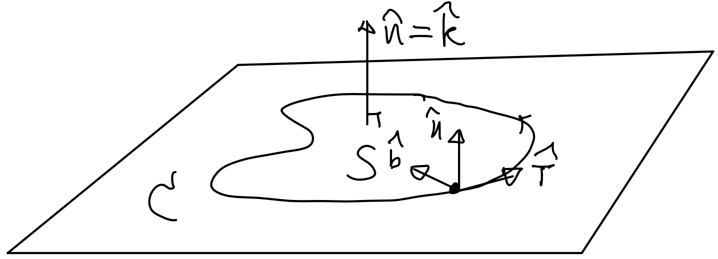
<u>eg 60</u>

(1)

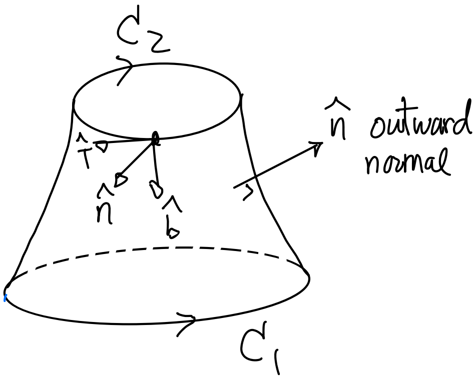


(2) $S \subset \mathbb{R}^2$ with $\hat{n} = \hat{k}$

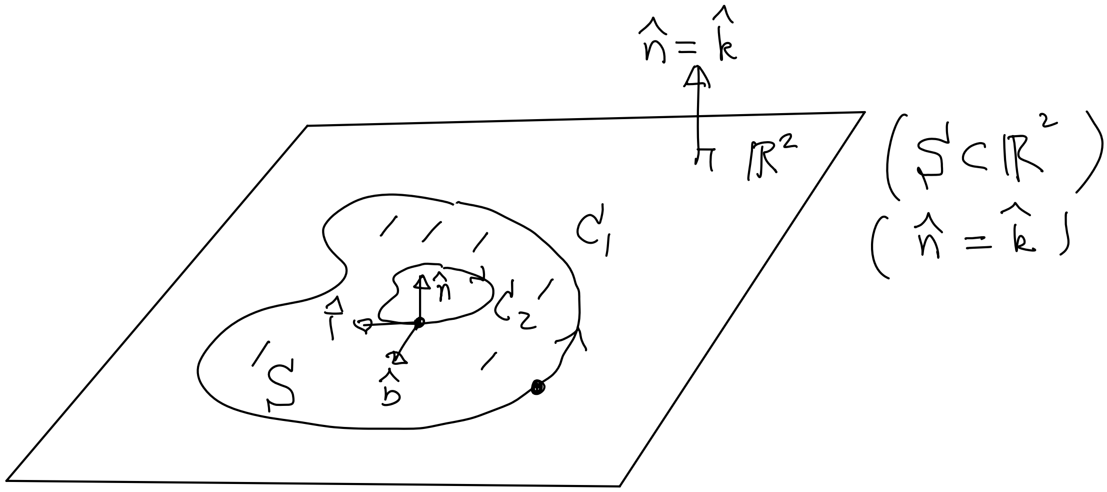
same as the usual anti-clockwise direction of a closed plane curve.



(3)



(4)



<u>Important remark</u>: If S is a region in $\mathbb{R}^2$, then a boundary component of S (C_1 or C_2 for instance) has "2" concepts of "oriented anti-clockwisely" with respect to

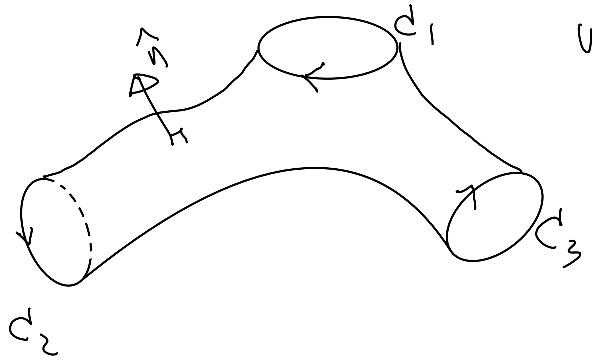
S = region and $\mathbb{R}^2$

Even S and $\mathbb{R}^2$ have the same orientation, i.e. $\hat{n} = \hat{k}$,

we still have the following situations: (C_1, C_2 as in figure)

	S (region)	$\mathbb{R}^2$
C_1	anti-clockwise (+)	anti-clockwise (+)
C_2	anti-clockwise (+)	clockwise (−)

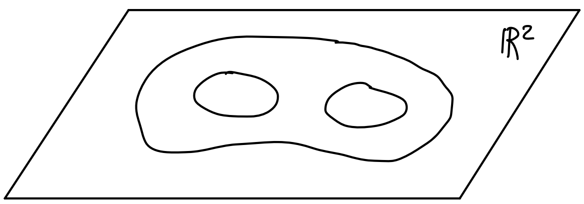
(5)



(Ex!)

what is the oriented of C_i s.t. their oriented anti-clockwise <u>with respect to</u> $\hat{n}$? (Ex!)

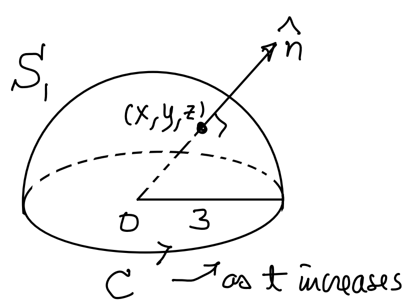
(<u>Similarly</u> <u>optional</u>:



?)

<u>eg 61</u> <u>Verifying Stokes' Thm</u>

(a) $S_1 : x^2+y^2+z^2 = 9,\ z \geq 0$



with <u>upward normal</u> $\hat{n}$ (i.e. $\hat{k}$-component > 0)

boundary $C : x^2+y^2 = 9,\ z = 0$

Parametrize C:

$\vec{r}(t) = \langle 3\cos t, 3\sin t, 0 \rangle$, $0 \leq t \leq 2\pi$

$= 3\cos t\,\hat{i} + 3\sin t\,\hat{j}$

(has the correct direction, i.e. oriented anti-clockwisely wrt $\hat{n}$)

Suppose $\vec{F} = y\hat{i} - x\hat{j}$

$$\oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} (3\sin t\,\hat{i} - 3\cos t\,\hat{j}) \cdot (-3\sin t\,\hat{i} + 3\cos t\,\hat{j})\, dt$$

$= -18\pi$ (check!)

For the surface integral:

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix} = -2\hat{k} \quad \text{(check!)}$$

Since S_1 is a hemisphere (upper) centered at origin of radius 3

$$\hat{n} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2+y^2+z^2}} = \frac{1}{3}(x\hat{i} + y\hat{j} + z\hat{k})$$

($z \geq 0 \Leftrightarrow$ upward)

The surface S_1 can be regarded as a level surface given by

$$g(x,y,z) = x^2+y^2+z^2 = 9$$

$$\Rightarrow \vec{\nabla} g = (2x, 2y, 2z)$$

Since $z > 0$ (except the boundary) on S_1, $\frac{\partial g}{\partial z} = 2z \neq 0$

Hence $d\sigma = \dfrac{|\vec{\nabla} g|}{\left|\frac{\partial g}{\partial z}\right|}\,dx\,dy = \dfrac{\sqrt{4x^2+4y^2+4z^2}}{|2z|}\,dx\,dy = \dfrac{3}{z}\,dx\,dy$

(since $z > 0$)

Therefore $\displaystyle\iint_{S_1} (\vec{\nabla}\times\vec{F})\cdot\hat{n}\,d\sigma$

$$= \iint_{\{x^2+y^2\leq 9\}} (-2\hat{k})\cdot\frac{1}{3}(x\hat{i}+y\hat{j}+z\hat{k})\ \frac{3}{z}\,dx\,dy$$

$$= \iint_{\{x^2+y^2\leq 9\}} (-2)\,dx\,dy = -18\pi \qquad \text{(check!)}$$