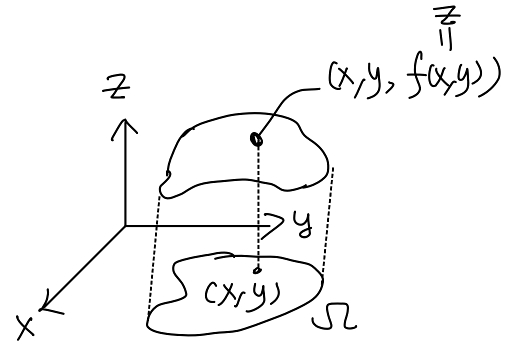
Surface area of a graph

$z = f(x,y)$, $(x,y) \in \Omega$



Choose the following "natural" parametrization of the graph

$$\vec{r}(x,y) = x\hat{i} + y\hat{j} + f(x,y)\hat{k}$$

$$\Rightarrow \begin{cases} \vec{r}_x = \hat{i} + f_x\hat{k} \\ \vec{r}_y = \hat{j} + f_y\hat{k} \end{cases}$$

$$\Rightarrow \quad \vec{r}_x \times \vec{r}_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = -f_x\hat{i} - f_y\hat{j} + \hat{k}$$

$$\Rightarrow \quad |\vec{r}_x \times \vec{r}_y| = \sqrt{f_x^2 + f_y^2 + 1} = \sqrt{1 + |\vec{\nabla} f|^2} \geq 1$$

(non-zero, hence "smooth" if $f \in C^1$)

<u>Thm 11</u>: The surface area of a C^1 graph S given by

$$z = f(x,y), \ (x,y) \in \Omega \subset \mathbb{R}^2$$

is

$$\text{Area}(S) = \iint_\Omega \sqrt{1 + |\vec{\nabla} f|^2}\, dA = \iint_\Omega \sqrt{1 + f_x^2 + f_y^2}\, dA$$

(Similarly for $x = f(y,z)$ or $y = f(x,z)$)

Implicit Surface (Level surface)

Suppose S is given by $F(x,y,z) = c$

i.e. $S = F^{-1}(c)$

(Note: F is a function of 3-variables, not vector field)

eg53: $F(x,y,z) = x^2+y^2+z^2$

Is $F^{-1}(0)$ a surface?

No, since $F^{-1}(0) = \{(0,0,0)\}$, not a surface!

Remark: If $\vec{\nabla} F \neq \vec{0}$ at a point, then IFT implies that

$S = F^{-1}(c)$ is a "surface" (c = value of F at that point)

near that point (in fact, a graph!)

eg53 (cont'd) $\vec{\nabla} F = 2x\,\hat{i} + 2y\,\hat{j} + 2z\,\hat{k}$

$\therefore \quad \vec{\nabla} F = \vec{0} \Leftrightarrow (x,y,z) = (0,0,0)$

Hence if $c>0$, then $\forall\, (x,y,z) \in F^{-1}(c)$, we have

$$\vec{\nabla} F(x,y,z) \neq \vec{0}$$

(since $x^2+y^2+z^2 = c > 0 \Rightarrow (x,y,z) \neq (0,0,0)$)

$\Rightarrow S = F^{-1}(c)$ $(\forall\, c>0)$ is a surface

(What are these surfaces?)

<u>Terminology</u> : $S = F^{-1}(c)$ is said to be <u>smooth</u>

if (1) F is C^1 on S, and

(2) $\vec{\nabla} F \neq \vec{0}$ on S.

How to compute surface area for a smooth level surface $S = F^{-1}(c)$?

By $\vec{\nabla} F \neq \vec{0}$, at least one of the partial derivatives F_x, F_y, & F_z is nonzero. Let assume $F_z = \frac{\partial F}{\partial z} \neq 0$ (the other cases are similar)

IFT $\Rightarrow$ $S = F^{-1}(c) = \{ F(x,y,z) = c \}$

can be written (locally) as a graph

$$z = f(x,y) \qquad \text{(near a point)}$$

i.e. $F(x, y, f(x,y)) = c$ (near a point)

Then chain rule $\Rightarrow$ $\begin{cases} f_x = -\dfrac{F_x}{F_z} \\ f_y = -\dfrac{F_y}{F_z} \end{cases}$ $(F_z \neq 0)$

Hence $\text{Area}(S) = \iint_\Omega \sqrt{1 + f_x^2 + f_y^2}\, dA$ where Ω = domain of the (local) $z = f(x,y)$.

$$= \iint_\Omega \sqrt{1 + \frac{F_x^2}{F_z^2} + \frac{F_y^2}{F_z^2}}\, dx\,dy$$

(↑ practically may be hard to find.)

$$= \iint_\Omega \frac{\sqrt{F_x^2 + F_y^2 + F_z^2}}{|F_z|}\, dx\,dy$$

Thm 12 If $S = F^{-1}(c)$ is a smooth level surface such that $F_z \neq 0$, and can be represented by an implicit function over a domain Ω.

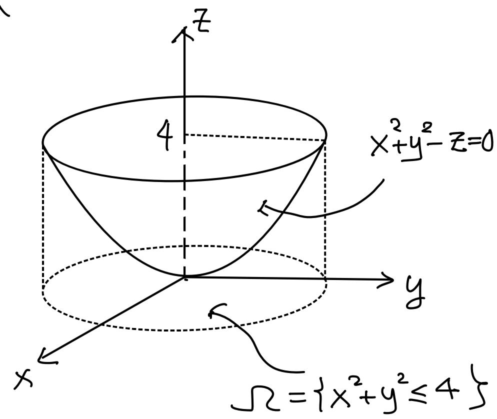
Then $$\text{Area}(S) = \iint_{\Omega} \frac{|\vec{\nabla} F|}{|F_z|}\, dA = \iint_{\Omega} \frac{|\vec{\nabla} F|}{|F_z|}\, dx\,dy$$

(Similar for the cases that $F_x \neq 0$ or $F_y \neq 0$)

eg 54: Find surface area of the paraboloid

$x^2+y^2-z=0$ below $z=4$

(This is in fact a graph, but we do it using method of level surface)



Soln: Let $F(x,y,z) = x^2+y^2-z$

For $z=4$, $x^2+y^2-z=0 \Rightarrow x^2+y^2=4$

$\Rightarrow$ projected region $\Omega = \{(x,y): x^2+y^2 \leq 4\}$

$$\vec{\nabla} F = 2x\hat{i} + 2y\hat{j} - \hat{k}$$

i.e. $F_z = -1 \neq 0$, $\forall (x,y) \in \Omega$.

$$\therefore \text{ Surface Area} = \iint_{\Omega} \frac{|\vec{\nabla} F|}{|F_z|}\, dA = \iint_{\Omega} \frac{\sqrt{4x^2+4y^2+1}}{|-1|}\, dx\,dy$$

(check!) $$= \iint_{\{x^2+y^2 \leq 4\}} \sqrt{4x^2+4y^2+1}\, dx\,dy = \frac{\pi}{6}\left[(\sqrt{17})^3 - 1\right]$$

(using polar coordinates) #

Def 16 Surface Integral (of a function)

Suppose $G: S \to \mathbb{R}$ is a continuous function on a surface S, parametrized by $\vec{r}(u,v)$, $(u,v) \in R$ (region R). Then the integral of G on S is

$$\iint_S G \, d\sigma \stackrel{\text{def}}{=\!=} \iint_R G(\vec{r}(u,v)) \, |\vec{r}_u \times \vec{r}_v| \, dA$$

area element of S

element area of the parameter space $dA = du\,dv$

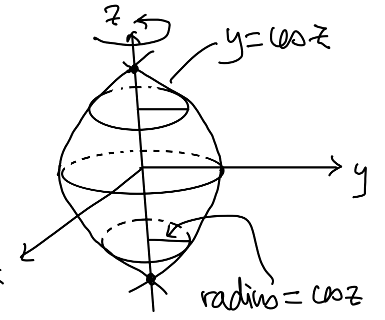
Note: In the cases of graph or level surface, we have

(i) $$\iint_S G \, d\sigma = \iint_{(x,y)} G(x,y,f(x,y)) \sqrt{1+|\vec{\nabla} f|^2} \, dx\,dy$$ (for $z=f(x,y)$)

(ii) $$\iint_S G \, d\sigma = \iint_{(x,y)} G(x,y,z) \frac{|\vec{\nabla} F|}{|F_z|} \, dx\,dy$$ (for $F(x,y,z)=c$, $F_z \neq 0$)

(may be difficult to find these: region & z in terms of (x,y))

eg 56 (a surface of revolution of the curve $y = \cos z$)



$\left(-\frac{\pi}{2} \le z \le \frac{\pi}{2}\right)$

Let $G(x,y,z) = \sqrt{1-x^2-y^2}$ be a function on S.

Find $\iint_S G \, d\sigma$.

Soln: S can be parametrized by

$$\begin{cases} x = \cos z \cos\theta & \theta \in [-\pi, \pi] \\ y = \cos z \sin\theta & z \in [-\frac{\pi}{2}, \frac{\pi}{2}] \\ z = z \end{cases}$$

i.e. $\vec{r}(\theta, z) = \cos z \cos\theta\,\hat{i} + \cos z \sin\theta\,\hat{j} + z\hat{k}$

(Note: there is an exceptional set of "1-dim" so that $\vec{r}$ is not one-to-one or not smooth corresponds to $\theta = \pi$ or $-\pi$, $z = -\frac{\pi}{2}$ or $\frac{\pi}{2}$.)

$$\Rightarrow \begin{cases} \vec{r}_\theta = -\cos z \sin\theta\,\hat{i} + \cos z \cos\theta\,\hat{j} \\ \vec{r}_z = -\sin z \cos\theta\,\hat{i} - \sin z \sin\theta\,\hat{j} + \hat{k} \end{cases}$$

$$\Rightarrow \quad \vec{r}_\theta \times \vec{r}_z = \cos z \cos\theta\,\hat{i} + \cos z \sin\theta\,\hat{j} + \sin z \cos z\,\hat{k} \quad \text{(Check!)}$$

$$\Rightarrow \quad |\vec{r}_\theta \times \vec{r}_z| = \sqrt{\cos^2 z(1+\sin^2 z)} = \cos z\sqrt{1+\sin^2 z} \quad \text{(Check!)}$$

$$(\cos z \geq 0 \text{ for } -\tfrac{\pi}{2} \leq z \leq \tfrac{\pi}{2})$$

Then
$$\iint_S G\,d\sigma = \int_{-\pi}^{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} G(\vec{r}(\theta,z))\,|\vec{r}_\theta \times \vec{r}_z|\,dz\,d\theta$$

$$= \int_{-\pi}^{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1-x^2-y^2}\,\cos z\sqrt{1+\sin^2 z}\,dz\,d\theta$$

$$= \int_{-\pi}^{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1-\cos^2 z}\,\cos z\sqrt{1+\sin^2 z}\,dz\,d\theta$$

(Check!) $$= 4\pi\int_0^{\frac{\pi}{2}} \sin z \cos z\sqrt{1+\sin^2 z}\,dz = \cdots = \frac{4\pi}{3}(2\sqrt{2}-1)$$

\#

Orientation of Surfaces

To integrate vector fields over surfaces, we need

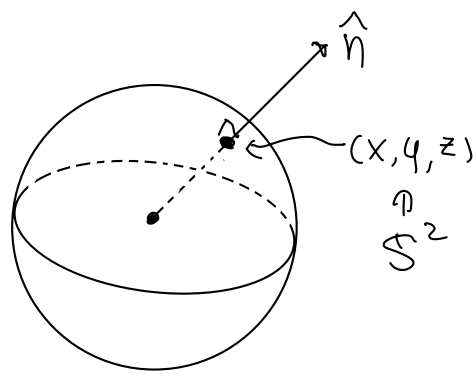
> **Def 17** (Orientation of a surface in $\mathbb{R}^3$)
> A surface S is orientable if one can define a unit normal vector field continuously at every point of S.

eg 57: (i) $S^2 = \{x^2+y^2+z^2=1\}$

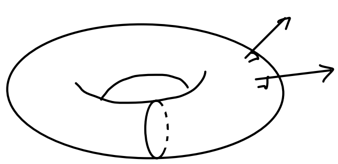
$\hat{n} = x\hat{i}+y\hat{j}+z\hat{k}$ is cts.

$(|\hat{n}| = \sqrt{x^2+y^2+z^2} = 1)$

$\therefore$ S^2 is orientable

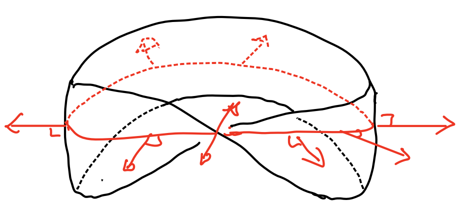


(ii)



Torus is orientable

(iii) Möbius strip is not orientable
(usually refered as a surface of one side)



Remark: Parametric surface $S = \vec{r}(u,v)$ are alway orientable:

the unit normal vector field $\hat{n} = \dfrac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$ given by

the parametrization is a continuous unit normal vector field on S.

($\vec{r}_u, \vec{r}_v$ "continuous" tangent vectors $\Rightarrow \vec{r}_u \times \vec{r}_v$ is a "continuous" normal vector

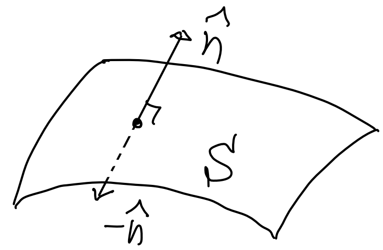
$|\vec{r}_u \times \vec{r}_v| \neq 0 \Rightarrow \dfrac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$ continuous unit normal vector field)

Terminology

Given a connected orientable surface $S \subset \mathbb{R}^3$, there are

two ways to assign the continuous unit normal vector field

Suppose S is orientable and we have already chosen one continuous unit normal vector field $\hat{n}$, before choosing a parametrization.



Ref/Def: We said that a parametrization $\vec{r}(u,v)$ of S is compatible with the orientation of S given by

the unit normal vector field $\hat{n} = \dfrac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$.

(The chosen unit normal vector field is also referred as the orientation of S)