Surface Area & Integral

Def 14 Parametric Surface (Surface with parametrization)

A parametric surface (or a parametrization of a surface) in $\mathbb{R}^3$ is a mapping of 2-variables into $\mathbb{R}^3$:

$$\vec{r}(u,v) = x(u,v)\,\hat{i} + y(u,v)\,\hat{j} + z(u,v)\,\hat{k}$$

And it is called smooth if

(1) $\vec{r}$ is C^1 (i.e. $x_u, x_v, y_u, y_v, z_u, z_v$ are continuous)

(2) $\boxed{\vec{r}_u \times \vec{r}_v \neq \vec{0}}$ $\forall\, u, v$

where

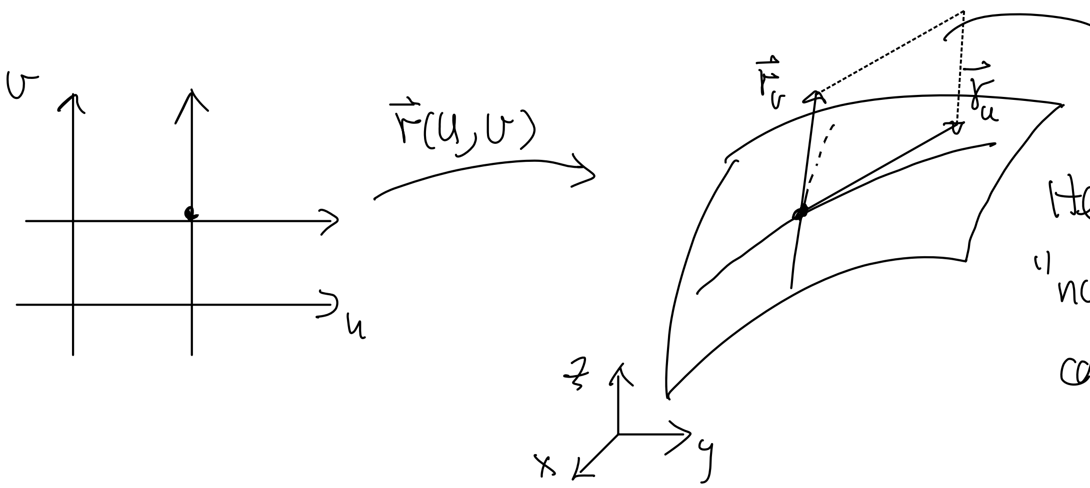
$$\begin{cases} \vec{r}_u = \dfrac{\partial \vec{r}}{\partial u} = \dfrac{\partial x}{\partial u}\hat{i} + \dfrac{\partial y}{\partial u}\hat{j} + \dfrac{\partial z}{\partial u}\hat{k} \\ \vec{r}_v = \dfrac{\partial \vec{r}}{\partial v} = \dfrac{\partial x}{\partial v}\hat{i} + \dfrac{\partial y}{\partial v}\hat{j} + \dfrac{\partial z}{\partial v}\hat{k} \end{cases}$$

$$\left(\begin{cases} \vec{r}_u = x_u\hat{i} + y_u\hat{j} + z_u\hat{k} \\ \vec{r}_v = x_v\hat{i} + y_v\hat{j} + z_v\hat{k} \end{cases} \right)$$

Note: Condition (2) $\Rightarrow$ $\vec{r}_u, \vec{r}_v$ are linearly independent

$\Rightarrow$ span$\{\vec{r}_u, \vec{r}_v\}$ is in fact a 2-dim'l subspace.

$\Rightarrow$ "surface" cannot be degenerated to a curve or a point.

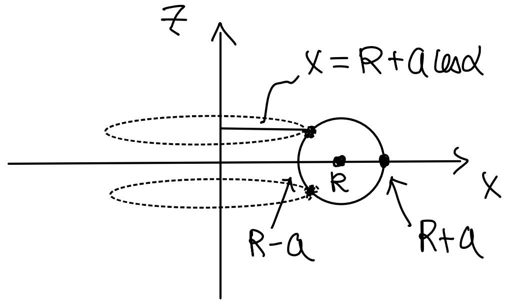


area $= |\vec{r}_u \times \vec{r}_v| \neq 0$

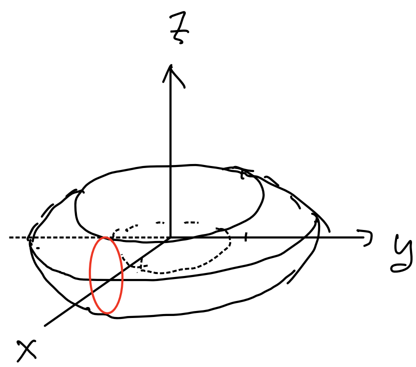
Hence, it is a "non-degenerate" condition.

<u>eg51</u> (Torus)

Consider the circle on the xz-plane (ie. $y=0$) with radius $a>0$ centered at $(x,z)=(R,0)$ with $R>a$.



A parametrization is
$$\begin{cases} x = R + a\cos\alpha \\ z = a\sin\alpha \end{cases} \qquad \alpha \in [0, 2\pi]$$



rotating the circle, we have a Torus

Then the parametrization of the Torus is
$$\begin{cases} x = (R+a\cos\alpha)\cos\theta \\ y = (R+a\sin\alpha)\sin\theta \\ z = a\sin\alpha \end{cases}, \qquad \begin{matrix} \alpha \in [0,2\pi] \\ \theta \in [0,2\pi] \end{matrix}$$

Ex: Check that it is a smooth surface:

It is clearly C^1, need to check

$$(x_\alpha, y_\alpha, z_\alpha) \times (x_\theta, y_\theta, z_\theta) \neq \vec{0}$$

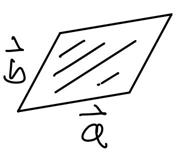
(See next example)

Note: This torus can also be described as

$$\left(\sqrt{x^2+y^2} - R\right)^2 + z^2 = a^2 \qquad \text{(Ex!)}$$

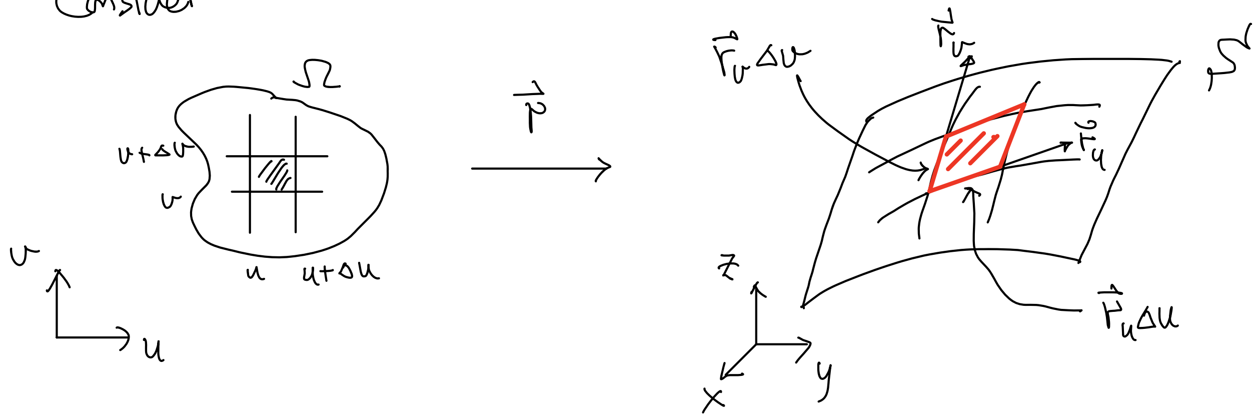
Surface Area

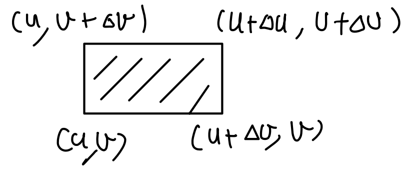
Recall: for $\vec{a}, \vec{b} \in \mathbb{R}^3$

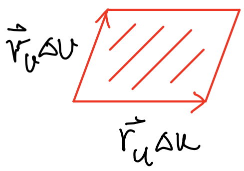
$|\vec{a} \times \vec{b}| = \text{Area}($  $)$

Let $\vec{r}(u,v)$ be a parametrization of a surface S with $(u,v) \in \Omega$

Consider



$\Rightarrow$ "Area" on the surface corresponding to 

is approx. $=$ Area ()

$$= \left| (\vec{r}_u \Delta u) \times (\vec{r}_v \Delta v) \right|$$

$$= |\vec{r}_u \times \vec{r}_v| \Delta u \Delta v$$

Hence "Area element" of S, denoted by $d\sigma$, is given by

$$d\sigma = |\vec{r}_u \times \vec{r}_v| \, du\, dv$$

$$\boxed{d\sigma = |\vec{r}_u \times \vec{r}_v| \, dA}$$

(dA ← area element in the (u,v)-space.)

Therefore, we make the following

Def 15: Let $S \subset \mathbb{R}^3$ be a smooth parametric surface given by $\vec{r}(u,v)$ for $(u,v) \in \Omega \subset \mathbb{R}^2$. Then

$$\text{Area}(S) \overset{\text{def}}{=\!=} \iint_{\Omega} |\vec{r}_u \times \vec{r}_v| \, dA$$

$$= \iint_{\Omega} \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| dA$$

(i.e. $\text{Area}(S) = \iint_{\Omega} d\sigma$)

eg 52: Surface area of torus given by ($R > a > 0$ are constants)

$$\begin{cases} x = (R + a\cos\alpha)\cos\theta \\ y = (R + a\sin\alpha)\sin\theta \\ z = a\sin\alpha \end{cases}, \quad \begin{matrix} \alpha \in [0, 2\pi] \\ \theta \in [0, 2\pi] \end{matrix}$$

i.e. $$\vec{r}(\alpha, \theta) = (R + a\cos\alpha)\cos\theta\, \hat{i} + (R + a\sin\alpha)\sin\theta\, \hat{j} + a\sin\alpha\, \hat{k}$$

$$\Rightarrow \begin{cases} \dfrac{\partial \vec{r}}{\partial \alpha} = -a\sin\alpha\cos\theta\, \hat{i} - a\sin\alpha\sin\theta\, \hat{j} + a\cos\alpha\, \hat{k} \\ \dfrac{\partial \vec{r}}{\partial \theta} = -(R + a\cos\alpha)\sin\theta\, \hat{i} + (R + a\cos\alpha)\cos\theta\, \hat{j} \end{cases}$$

& $$\frac{\partial \vec{r}}{\partial \alpha} \times \frac{\partial \vec{r}}{\partial \theta} = -a(R + a\cos\alpha)\cos\theta\cos\alpha\, \hat{i} - a(R + a\cos\alpha)\sin\theta\cos\alpha\, \hat{j} - (R + a\cos\alpha)\sin\alpha\, \hat{k}$$ (check!)

$$\left| \frac{\partial \vec{r}}{\partial \alpha} \times \frac{\partial \vec{r}}{\partial \theta} \right| = a(R + a\cos\alpha) \quad (>0)$$ (Check!)

(so, the surface is "smooth")

$$\text{Area (Torus)} = \iint_{\Omega} \left| \frac{\partial \vec{r}}{\partial \alpha} \times \frac{\partial \vec{r}}{\partial \theta} \right| dA$$

$$= \int_0^{2\pi} \int_0^{2\pi} a(R + a\cos\alpha)\, d\alpha\, d\theta$$

$$= 4\pi^2 R a$$ (check!) #