(Cont'd)

On the other hand, Fubini's Thm $\Rightarrow$

$$\iint_R -\frac{\partial M}{\partial y}\, dA = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} -\frac{\partial M}{\partial y}\, dy \right) dx$$

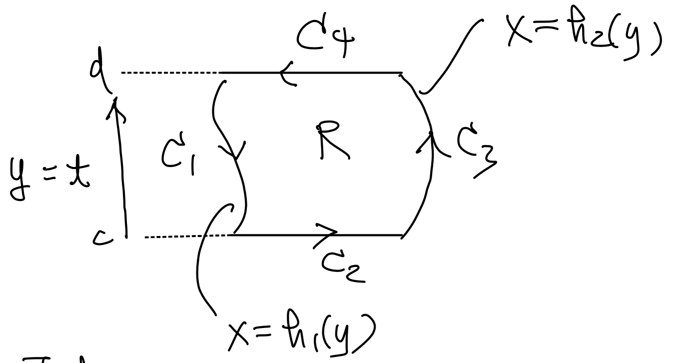
$$= \int_a^b -[M(x, g_2(x)) - M(x, g_1(x))]\, dx = \oint_{\partial R} M\, dx$$

Similar, R is also typo (2), R can be written as

$$R = \{ (x,y) : h_1(y) \le x \le h_2(y),\ c \le y \le d \}$$

$$\oint_{\partial R} N\, dy = -\int_c^d N(h_1(t), t)\, dt + 0$$

$$+ \int_c^d N(h_2(t), t)\, dt + 0$$



$$= \int_c^d [N(h_2(t), t) - N(h_1(t), t)]\, dt$$

$$= \int_c^d [N(h_2(y), y) - N(h_1(y), y)]\, dy$$

$$= \int_c^d \left(\int_{h_1(y)}^{h_2(y)} \frac{\partial N}{\partial x}\, dx \right) dy = \iint_R \frac{\partial N}{\partial x}\, dA$$

All together

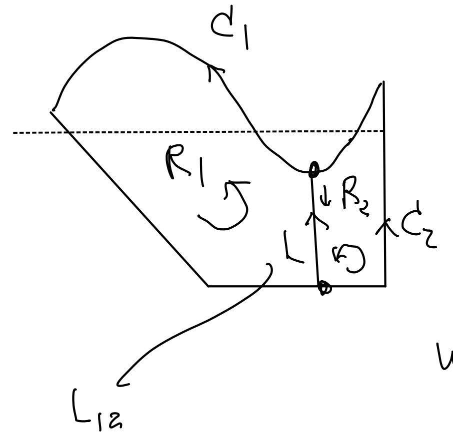
$$\oint_{\partial R} (M\,dx + N\,dy) = \iint_R -\frac{\partial M}{\partial y}\, dA + \iint_R \frac{\partial N}{\partial x}\, dA = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

#

Proof of Green's Thm for

R = finite union of simple regions with intersections only along some boundary line segments, and those line segments touch only at the end points at most.

eg:



R_1, R_2 = simple
but $R = R_1 \cup R_2 \neq$ simple

$\partial R_1 = C_1 + L$

$\partial R_2 = C_2 - L$

L

with anti-clockwise orientation

and $\partial R = C_1 + C_2$

By assumption $R = \cup R_i$ finite union s.t.

- R_i are simple, and
- $R_i \cap R_j$ = line segment of a common boundary portion denoted by L_{ij} $(i \neq j)$ (may be empty)

Then

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \sum_i \iint_{R_i} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

$$= \sum_i \oint_{\partial R_i} M dx + N dy \qquad \begin{pmatrix}\text{by Green's Thm} \\ \text{for simple region}\end{pmatrix}$$

Denote C_i = the part of ∂R_i with no intersection with any other R_j (except at the end points)

Then $$\partial R_i = C_i + \sum_{\substack{j \\ (j\neq i)}} L_{ij}$$

where L_{ij} is oriented according to the anti-clockwise orientation of ∂R_i

Hence $$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \sum_i \oint_{C_i + \sum_{\substack{j \\ (j\neq i)}} L_{ij}} M dx + N dy$$

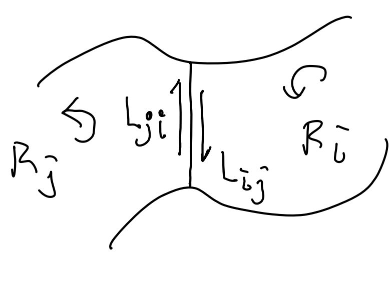
$$= \sum_i \oint_{C_i} M dx + N dy + \sum_i \int_{\sum_{\substack{j \\ (j\neq i)}} L_{ij}} M dx + N dy$$

Note that, as C_i is not a common boundary of any other R_j,

$$\sum_i C_i = \partial R$$

$$\therefore \quad \sum_i \oint_{C_i} M dx + N dy = \oint_{\partial R} M dx + N dy$$

Finally, we have $L_{ji} = -L_{ij}$



as R_i & R_j are located on the two different sides of the common boundary.

$$\sum_{i} \int_{\substack{\sum_j L_{ij} \\ (j \neq i)}} M\,dx + N\,dy = \sum_{i} \sum_{\substack{j \\ (j\neq i)}} \int_{L_{ij}} M\,dx + N\,dy$$

$$= \sum_{\substack{i,j \\ (i \neq j)}} \int_{L_{ij}} M\,dx + N\,dy$$

$$= \sum_{i<j} \int_{L_{ij}} M\,dx + N\,dy + \sum_{\substack{\cancel{j}<\cancel{i} \\ i<j}} \int_{L_{\cancel{ij}}\ ji} M\,dx + N\,dy \qquad \text{(changing dummy indexes)}$$

$$= \sum_{i<j} \left(\int_{L_{ij}} M\,dx + N\,dy + \int_{L_{ji}} M\,dx + N\,dy \right)$$

$$= \sum_{i<j} \left(\int_{L_{ij}} M\,dx + N\,dy + \int_{-L_{ij}} M\,dx + N\,dy \right)$$

$$= 0.$$

This 2^{nd} case basically include almost all situations in the level of Advanced Calculus.

The proof of <u>general case</u> needs "analysis" and will be omitted here.

※

Def 12: The *divergence* of $\vec{F} = M\hat{i} + N\hat{j}$ is defined to be

$$\operatorname{div}\vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$$

Note: $$\operatorname{div}\vec{F} = \lim_{\varepsilon\to 0} \frac{1}{\text{Area}(\bar{D}_\varepsilon(x,y))} \iint_{\bar{D}_\varepsilon(x,y)} \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA$$

$$= \lim_{\varepsilon\to 0} \frac{1}{\text{Area}(\bar{D}_\varepsilon(x,y))} \oint_{\partial\bar{D}_\varepsilon(x,y)} \vec{F}\cdot\hat{n}\, ds$$

$\underset{=}{\text{called}}$ "flux density".

Notation: For $f(x,y)$, $\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$ (gradient)

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y}\right) f$$

It is convenient to denote

$$\vec{\nabla} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y}\right)$$

Then $$\vec{\nabla}\cdot\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y}\right)\cdot\left(M\hat{i} + N\hat{j}\right)$$

$$= \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = \operatorname{div}\vec{F}$$

Hence we also write

$$\boxed{\operatorname{div}\vec{F} = \vec{\nabla}\cdot\vec{F}}$$

$\underline{\text{Def 13}}$: Define rot $\vec{F}$ to be

$$\boxed{\text{rot}\,\vec{F} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \qquad (\text{for } \vec{F} = M\hat{i} + N\hat{j})}$$

Note: $\text{rot}\,\vec{F} = \lim_{\varepsilon \to 0} \frac{1}{\text{Area}(\bar{D}_\varepsilon(x,y))} \iint_{\bar{D}_\varepsilon(x,y)} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$

$$= \lim_{\varepsilon \to 0} \frac{1}{\text{Area}(\bar{D}_\varepsilon(x,y))} \oint_{\partial \bar{D}_\varepsilon(x,y)} \vec{F} \cdot \hat{T}\, ds$$

$\overset{\text{called}}{=}$ circulation density

Using $\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y}$, we can write

$$\boxed{\text{rot}\,\vec{F} = (\vec{\nabla} \times \vec{F}) \cdot \hat{k}}$$

Since $\vec{F} = M\hat{i} + N\hat{j} + \underset{\text{number zero}}{0} \cdot \hat{k}$ (in $\mathbb{R}^3$) $(M = M(x,y)$ & $N = N(x,y))$

$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$ (in $\mathbb{R}^3$) $\left(\frac{\partial M}{\partial z} = \frac{\partial N}{\partial z} = 0\right)$

$$\Rightarrow \vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & 0 \end{vmatrix} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ M & N \end{vmatrix} \hat{k} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\hat{k}$$

(check!)

$\Rightarrow$ $\text{rot}\,\vec{F} = (\vec{\nabla} \times \vec{F}) \cdot \hat{k}$ i.e. $\hat{k}$-component of $\underline{\vec{\nabla} \times \vec{F}}$.

A name for $\vec{\nabla} \times \vec{F}$ is $\underline{\text{curl}\,\vec{F}}$: $\boxed{\text{curl}\,\vec{F} \overset{\text{def}}{=} \vec{\nabla} \times \vec{F}}$

In these notation, the Green's thm can be written as

<u>Vector form of Green's Thm</u>

<u>normal form</u>

$$\boxed{\oint_C \vec{F}\cdot\hat{n}\,ds = \iint_R \operatorname{div}\vec{F}\,dA}$$

or

$$\boxed{\oint_C \vec{F}\cdot\hat{n}\,ds = \iint_R \vec{\nabla}\cdot\vec{F}\,dA}$$

<u>tangential form</u>

$$\boxed{\oint_C \vec{F}\cdot\hat{T}\,ds = \iint_R \operatorname{curl}\vec{F}\cdot\hat{k}\,dA}$$

or

$$\boxed{\oint_C \vec{F}\cdot\hat{T}\,ds = \iint_R (\vec{\nabla}\times\vec{F})\cdot\hat{k}\,dA}$$

And Thm 10 can be written as

<u>Thm 10'</u>: Ω simply-connected & connected, $\vec{F}\in C^1$.

Then $\vec{F}$ = conservative $\iff$ $\operatorname{curl}\vec{F} = \vec{\nabla}\times\vec{F} = 0$

(check)

<u>Note</u>: (i) $\operatorname{curl}\vec{F} = \vec{\nabla}\times\vec{F}$ defined only in $\mathbb{R}^3$ $(\supset \mathbb{R}^2)$

(ii) but $\operatorname{div}\vec{F} = \vec{\nabla}\cdot\vec{F}$ can be defined on $\mathbb{R}^n$ for any n

In particular, in $\mathbb{R}^3$

<u>Def 12'</u> The <u>divergence</u> of $\vec{F} = M\hat{i} + N\hat{j} + L\hat{k}$ is defined to be

$$\operatorname{div}\vec{F} = \vec{\nabla}\cdot\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(M\hat{i} + N\hat{j} + L\hat{k}\right) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial L}{\partial z}$$

Then one can easily check the following facts: (Ex!)

$$\boxed{\begin{array}{l} \text{(i) } \vec{\nabla}\times(\vec{\nabla} f) = 0 \quad (\text{i.e. curl } \vec{\nabla} f = 0) \\ \text{(ii) } \vec{F} \text{ conservative} \Rightarrow \text{curl } \vec{F} = \vec{\nabla}\times\vec{F} = 0 \\ \text{(iii) } \vec{\nabla}\cdot(\vec{\nabla}\times\vec{F}) = 0 \quad (\text{i.e. div}(\text{curl } \vec{F}) = 0) \end{array}}$$

<u>Remark</u>: $\vec{\nabla}\cdot(\vec{\nabla} f) \neq 0$ in general, and it is called the <u>Laplacian</u> of f and is denoted by

$$\boxed{\begin{aligned} \vec{\nabla}^2 f &= \vec{\nabla}\cdot(\vec{\nabla} f) = \text{div}(\vec{\nabla} f) \\ &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \end{aligned}}$$

[In graduate level, it will be denoted by $\Delta = \vec{\nabla}^2$ or $\Delta = -\vec{\nabla}^2$]

The "operator" $\vec{\nabla}^2$ is called the <u>Laplace operator</u> and the equation $\vec{\nabla}^2 f = 0$ is called the <u>Laplace equation</u>. Solutions to the Laplace equation are called <u>harmonic functions</u>.

Pf of Thm 10 $(n=2)$

We only need to show that if Ω simply-connected (& connected) and

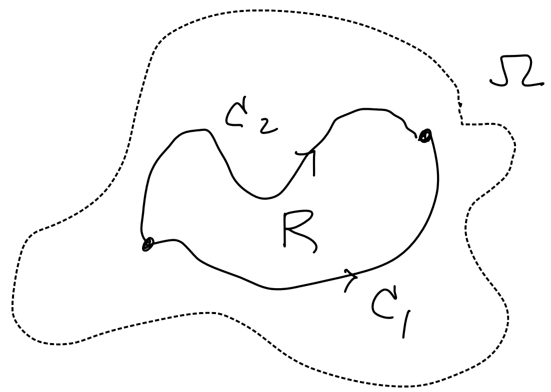
$$\vec{\nabla}\times\vec{F}=\vec{0} \quad \left(\text{i.e. } \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}\right)$$

then $\vec{F}$ is conservative.

Case 1 : C_1, C_2 have no intersection. (except the same end points)

Then "Ω is simply-connected"

$\Rightarrow$ the region R enclosed by C_1 and C_2 lies completely inside Ω.



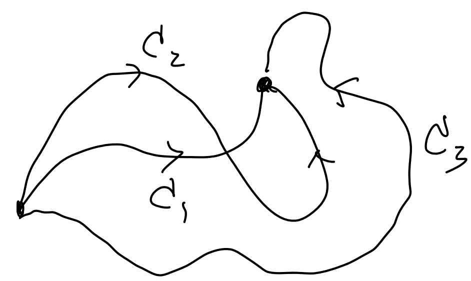
Then by Green's Thm

$$0=\iint_R\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dA=\pm\left(\int_{C_1}-\int_{C_2}\right)(Mdx+Ndy)$$

$$\Rightarrow \quad \int_{C_1} Mdx+Ndy=\int_{C_2} Mdx+Ndy$$

Case 2 C_1, C_2 intersect

Pick another curve C_3 with the same starting point and end point, and does not intersect C_1 or C_2.



Then by case 1, $\int_{C_1} M\,dx+N\,dy = \int_{C_3'} M\,dx+N\,dy = \int_{C_2'} M\,dx+N\,dy$

$\therefore$ $\int_C \vec{F}\cdot d\vec{r}$ is independent of the path and hence $\vec{F}$ is conservative. ※

In order to apply Green's Thm to more general situations, we need a <u>general form of Green's Thm:</u>

Suppose that we have a simple closed curve C' in $\mathbb{R}^2$

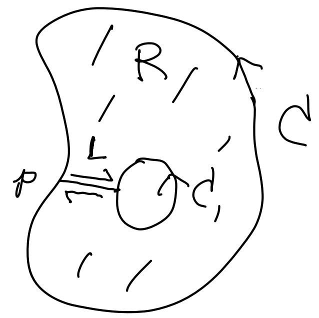
Suppose that $C_1, C_2, \cdots, C_n$ be pairwise disjoint, piecewise smooth, simple closed curves, such that $C_1, \cdots, C_n$ are enclosed by C'.
(All C', $C_1, \cdots, C_n$ are anti-clockwise oriented.)

Let R be the region between C' and $C_1, \cdots, C_n$.
Suppose that $\vec{F} = M\hat{i}+N\hat{j}$ is defined on some open set containing R, and is C^1. Then

$$\boxed{\iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) dA = \oint_{C'} M\,dx+N\,dy - \sum_{i=1}^{n} \oint_{C_i} M\,dx+N\,dy}$$

(This is the tangential form. The normal form is similar)

Sketch of Proof



For simplicity, only one C_1 inside C

We connect C & C_1 by an "arc" L

and consider the "simple" closed curve

(starting from p): $C^* = C + L - C_1 - L$

Then the region R enclosed between C & C_1 is the region

enclosed by C^* except the arc L.

$$\text{Hence } \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \iint_{R \setminus L} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

$$\overset{\text{Green's}}{=} \oint_{C^*} M\,dx + N\,dy$$

$$= \left(\oint_C + \int_L + \oint_{-C_1} + \int_{-L}\right)(M\,dx + N\,dy)$$

$$= \left(\oint_C + \int_L - \oint_{C_1} - \int_L\right)(M\,dx + N\,dy)$$

$$= \oint_C M\,dx + N\,dy - \oint_{C_1} M\,dx + N\,dy$$

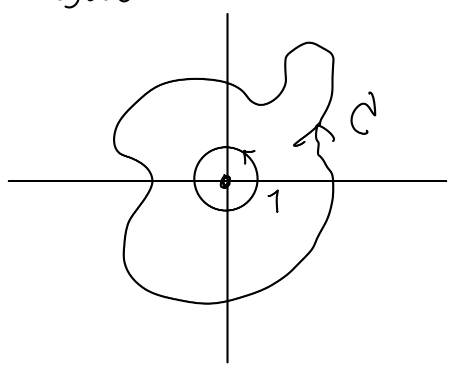
$\#$

eg 49: $\vec{F} = \dfrac{-y}{x^2+y^2}\hat{i} + \dfrac{x}{x^2+y^2}\hat{j}$ on $\mathbb{R}^2 \setminus \{(0,0)\} = \Omega$

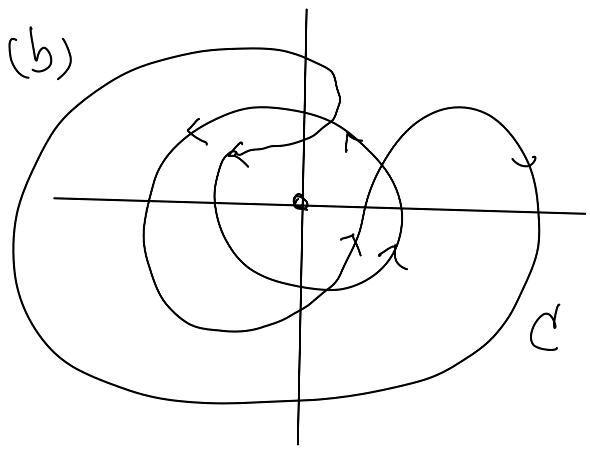
we've calculated $\oint_{C_1} \vec{F} \cdot d\vec{r} = 2\pi$ for $C_1 : x^2+y^2=1$ (anti-clockwise)

How about

(a)



(b)



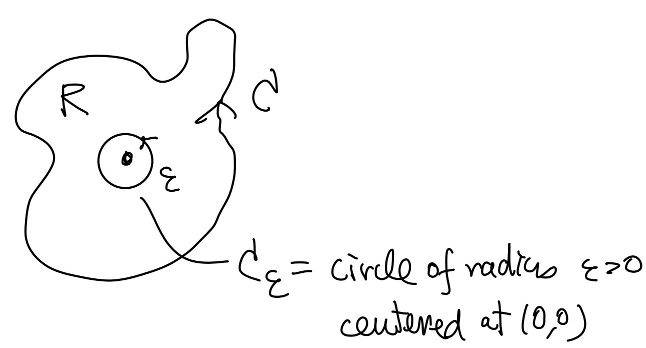
$\oint_C \vec{F} \cdot d\vec{r} = ?$

<u>Soln</u>

(a) Recall that $\vec{\nabla} \times \vec{F} = 0$

(Green's Thm doesn't apply to get $\oint_C \vec{F} \cdot d\vec{r} = 0$, since C encloses the origin $(0,0)$ where $\vec{F}$ is not defined)

Choose $\varepsilon > 0$ small enough such that the circle C_ε of radius ε centered at $(0,0)$ is completely enclosed by C.



$\vec{F}$ is smooth in the region R between C and C_ε.

Hence the general form of Green's Thm applied:

$$0 = \iint_R (\vec{\nabla}\times\vec{F})\cdot\hat{k}\, dA = \oint_C \vec{F}\cdot d\vec{r} - \oint_{C_\varepsilon} \vec{F}\cdot d\vec{r}$$

$$\Rightarrow \oint_C \vec{F}\cdot d\vec{r} = \oint_{C_\varepsilon} \vec{F}\cdot d\vec{r}$$

$$= \oint_{C_\varepsilon} \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$

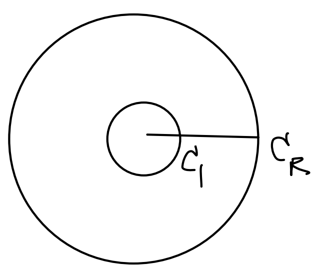
Parametrize C_ε by $\begin{cases} x = \varepsilon\cos\theta \\ y = \varepsilon\sin\theta \end{cases}$, $0 \le \theta \le 2\pi$

$$\Rightarrow \oint_{C_\varepsilon} \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \left[\frac{-\varepsilon\sin\theta}{\varepsilon^2}(-\varepsilon\sin\theta) + \frac{\varepsilon\cos\theta}{\varepsilon^2}(\varepsilon\cos\theta) \right] d\theta$$

$$= 2\pi$$

$$\Rightarrow \oint_C \vec{F}\cdot d\vec{r} = 2\pi \quad ※$$

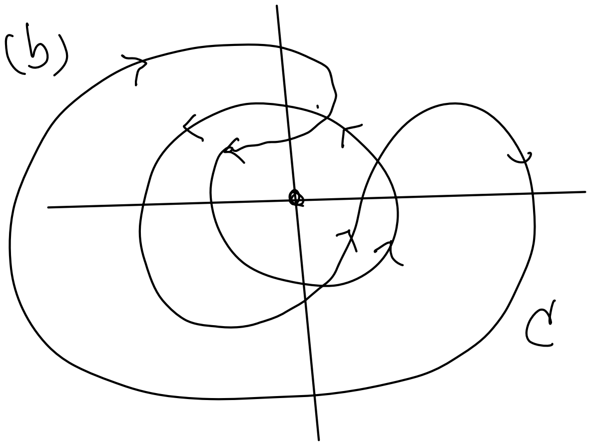
[In fact, we've proved that $\oint_{C_R} \vec{F}\cdot d\vec{r} = 2\pi$, $\forall$ radius $R>0$, which can be seen by consider the domain between C_1 & C_R



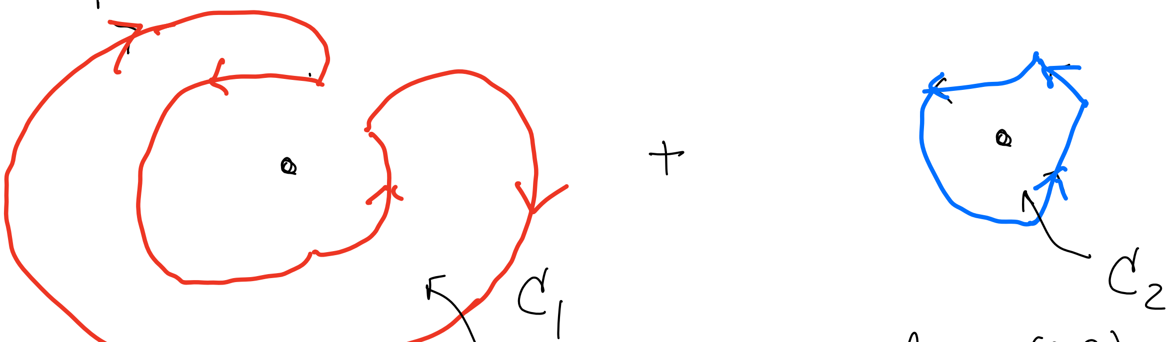
Green's Thm

$$\Rightarrow \oint_{C_1} \vec{F}\cdot d\vec{r} = \oint_{C_R} \vec{F}\cdot d\vec{r}$$
]

(b)



Decompose the curve into



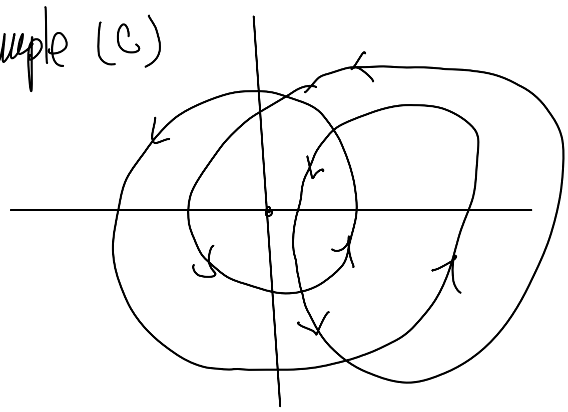
doesn't enclosed $(0,0)$

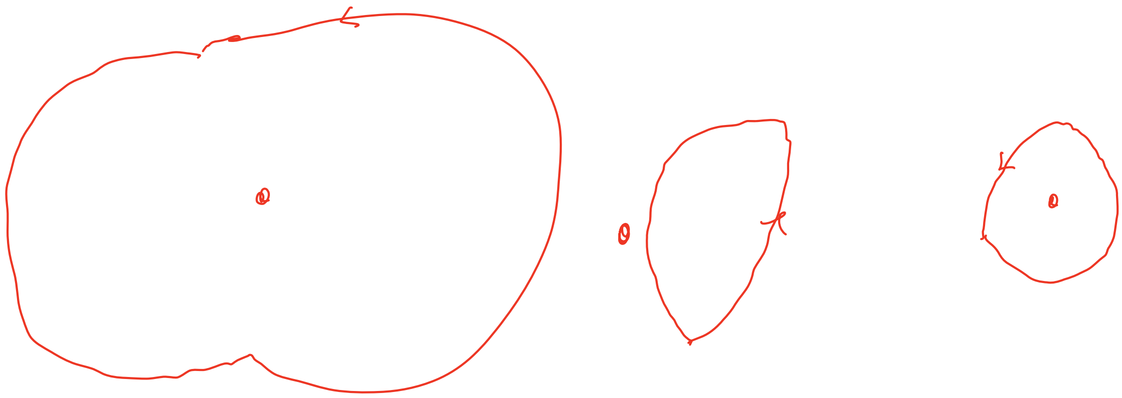
enclosing $(0,0)$ (once)

$\Rightarrow \quad \oint_{C_1} \vec{F}\cdot d r = 0 \qquad$ by part (a) $\oint_{C_2} \vec{F}\cdot dr = 2\pi$

$\therefore \quad \oint_{C} \vec{F}\cdot d\vec{r} = 0 + 2\pi = 2\pi$.

Addition example (c)





$\oint \cdots = 2\pi$ $\quad\quad$ $\oint \cdots = 0$ $\quad\quad$ $\oint \cdots = 2\pi$

Hence $\quad \oint_C \vec{F}\cdot d\vec{r} = 2\pi + 0 + 2\pi = 4\pi$

(optional ex! : think of some examples with -2π)