Remark: To prove Thm 10 in $\mathbb{R}^2$, we need the Green's Thm

(in $\mathbb{R}^3$, we need the Stokes' Thm)

Thm 11 (Green's Theorem)

Let $\Omega \subseteq \mathbb{R}^2$ be open, $\vec{F} = M\hat{i} + N\hat{j}$ be C^1 vector field on Ω;

C be a piecewise "smooth" simple closed anti-clockwise oriented curve enclosing a region R which lies entirely in Ω.

Then

- Normal Form

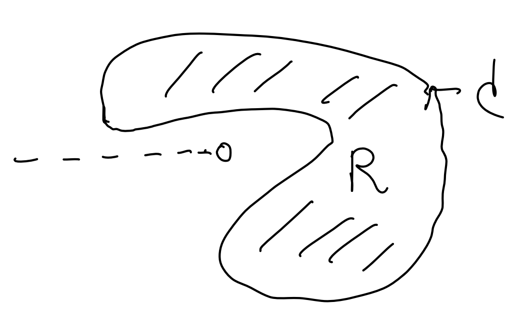
$$\oint_C \vec{F}\cdot\hat{n}\,ds = \boxed{\oint_C M\,dy - N\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy}$$

- Tangential Form

$$\oint_C \vec{F}\cdot\hat{T}\,ds = \boxed{\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy}$$

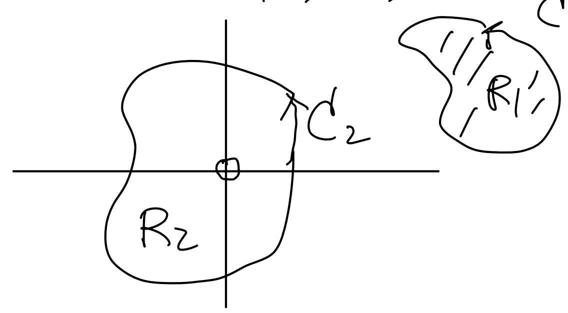
(Remark: The two forms are equivalent)

Note: $\Omega_1 = \mathbb{R}^2 \setminus \{x \le 0\}$



Green's Thm applies, since $R \subset \Omega_1$

$\Omega_2 = \mathbb{R}^2 \setminus \{(0,0)\}$



Green's Thm applies, since $R_1 \subset \Omega_2$

$(0,0) \in R_2$, but $(0,0) \notin \Omega_2$

$\Rightarrow R_2 \not\subset \Omega_2$,

Green's Thm doesn't apply.

eg 48 Verify both forms of Green's Thm for

$$\vec{F}(x,y) = (x-y)\hat{i} + x\hat{j} \quad \text{on } \Omega = \mathbb{R}^2, \text{ is } C^{\infty}.$$

$$C = \text{unit circle} = \vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j}, \ t \in [0, 2\pi]$$

Then R = region enclosed by $C = \{x^2+y^2<1\}$ the unit disc.

(We also write $C = \partial R$ boundary of R)

Soln $\quad M = x-y, \quad N = x$

$$\frac{\partial M}{\partial x} = 1, \quad \frac{\partial M}{\partial y} = -1 \quad ; \quad \frac{\partial N}{\partial x} = 1, \quad \frac{\partial N}{\partial y} = 0$$

On C, $x = \cos t$, $y = \sin t$, $t \in [0, 2\pi]$

Normal form $\quad$ L.H.S. $= \oint_C M\,dy - N\,dx$

$$= \int_0^{2\pi} (\cos t - \sin t)\,d\sin t - \cos t\,d\cos t$$

$$= \int_0^{2\pi} \cos^2 t\,dt = \pi \quad \text{(check!)}$$

$$\text{RHS} = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy = \iint_R (1+0)\,dx\,dy = \pi$$

Tangential form $\quad$ L.H.S. $= \oint_C M\,dx + N\,dy = 2\pi$ (check!)

$$\text{R.H.S.} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R (1-(-1))\,dx\,dy = 2\pi$$

(Note: This example shows that even the 2 forms are equivalent, the values involved may differ.)

Pf of Green's Thm (tangential form)

Recall: A region R is of special type:

type (1): If $R = \{(x,y) : a \leq x \leq b,\ g_1(x) \leq y \leq g_2(x)\}$

for some continuous functions $g_1(x)$ & $g_2(x)$.

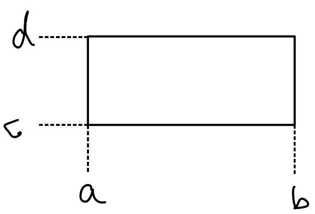
type (2): If $R = \{(x,y) : h_1(y) \leq x \leq h_2(y),\ c \leq y \leq d\}$

for some continuous functions $h_1(x)$ & $h_2(x)$.

Now: If R is both type (1) and type (2), it said

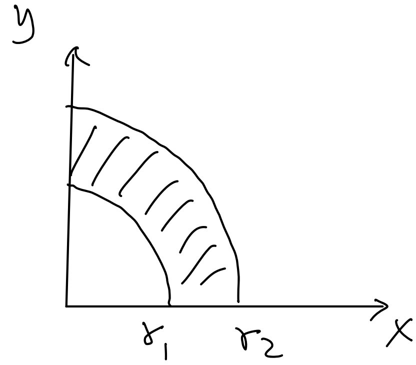
to be simple.

eg48: (i)



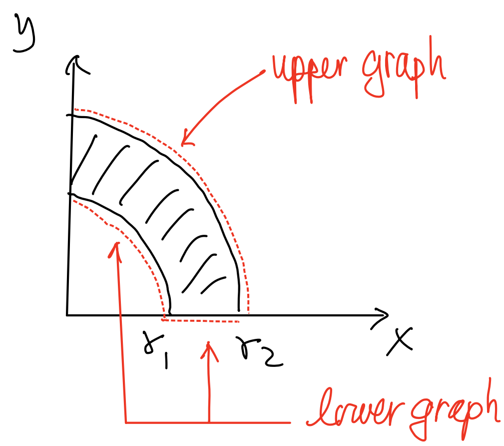
rectangle is simple

(ii)



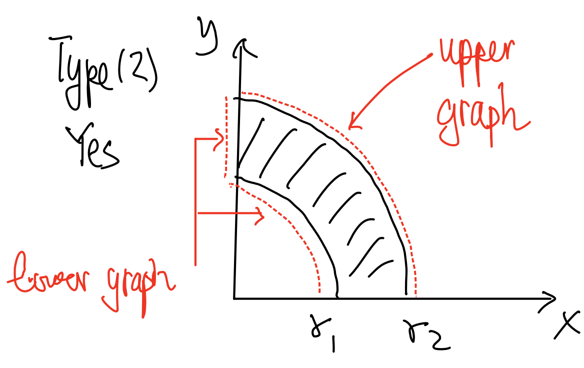
Type (1)

Yes



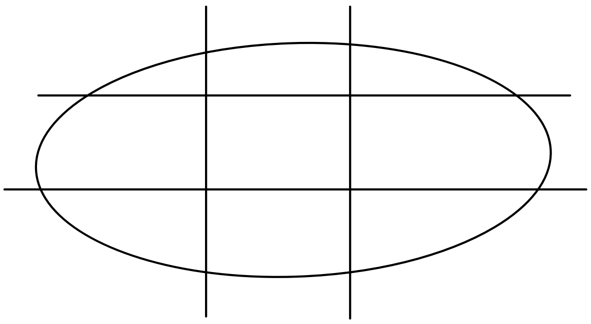
Type (2)

Yes



$\Rightarrow$ simple region

(iii')



2 intersections at most

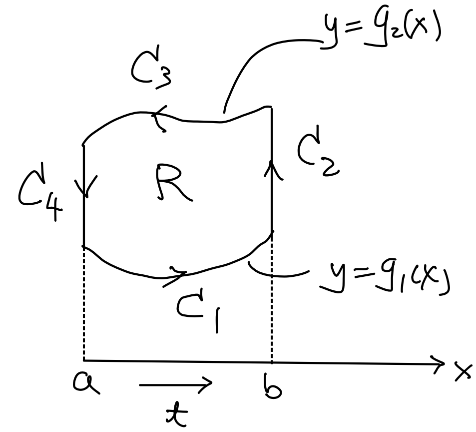
2 intersections at most

$$\forall a \in \mathbb{R}: \left.\begin{array}{l} \#\{\partial R \cap \{x=a\}\} \le 2 \\ \#\{\partial R \cap \{y=a\}\} \le 2 \end{array}\right\} \Rightarrow \underline{\text{simple}}$$

(provided ∂R is piecewise smooth).

(Proof: omitted)

<u>Pf of Green's Thm for Simple Region</u>

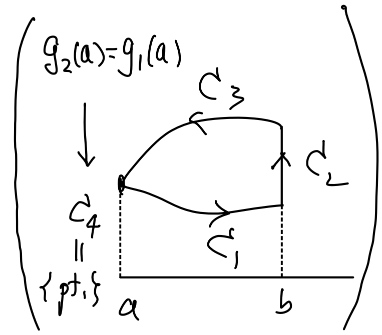


By definition, R is of type (1) and can be written as

$$R = \{(x,y) : a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$$

Let denote the components of the boundary of R by C_1, C_2, C_3, and C_4 as in the figure

(Note: C_2 and/or C_4 could just be a point)



Then $\partial R = C_1 + C_2 + C_3 + C_4$ as oriented curve

(using "+" instead of "$\cup$" to denote the orientation)

Now $C_1 = \{y = g_1(x)\}$ can be parametrized by

(x,y): $\vec{r}(t) = (t, g_1(t))$, $a \le t \le b$ (with correct orientation)

$$\therefore \int_{C_1} M\,dx = \int_a^b M(t, g_1(t))\,dt$$

Similarly "$-C_3$" can be parametrized by

$\vec{r}(t) = (t, g_2(t))$, $a \le t \le b$ (with correct orientation)

$$\therefore \int_{-C_3} M\,dx = \int_a^b M(t, g_2(t))\,dt$$

$$\Rightarrow \int_{C_3} M\,dx = -\int_{-C_3} M\,dx = -\int_a^b M(t, g_2(t))\,dt$$

For $C_2 = \{x = b\}$, it can be parametrized by

$\vec{r}(t) = (b, t)$, $g_1(b) \le t \le g_2(b)$ (with correct orientation)

$$\Rightarrow \int_{C_2} M\,dx = 0 \quad \left(\text{since } \frac{dx}{dt} \equiv 0\right)$$

Similarly $\int_{C_4} M\,dx = -\int_{-C_4} M\,dx = 0$

Hence $$\oint_{\partial R} M\,dx = \sum_{i=1}^{4} \int_{C_i} M\,dx$$

$$= \int_a^b \left[M(t, g_1(t)) - M(t, g_2(t))\right] dt$$

$$\left(= \int_a^b \left[M(x, g_1(x)) - M(x, g_2(x))\right] dx\right)$$

(To be cont'd)