Def 11 A vector field is defined to be continuous / differentiable / C^k if the component functions are.

eg38:

$\vec{F}(x,y) = \vec{r}(x,y) = x\hat{i} + y\hat{j}$ is C^∞ (position vector)

$\vec{F}(x,y) = \dfrac{-y\hat{i} + x\hat{j}}{\sqrt{x^2+y^2}}$ is not continuous in $\mathbb{R}^2$ (but continuous in $\mathbb{R}^2 \setminus \{(0,0)\}$)

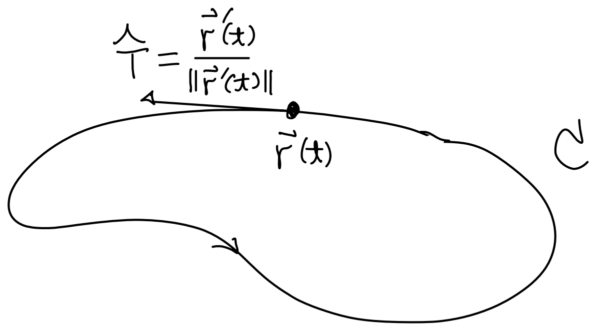
Line integral of vector field

Def 12: Let C be a curve with "orientation" given by a parametrization $\vec{r}(t)$ with $\vec{r}\,'(t) \neq 0$, $\forall t$. Define the line integral of a vector field $\vec{F}$ along C to be

$$\boxed{\int_C \vec{F} \cdot \hat{T}\, ds}$$

where $\hat{T} = \dfrac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$ is the unit tangent vector field along C.

(i.e. C is oriented in the direction of $\vec{r}\,'(t)$ or $\hat{T}$ at every point)



Note: If $\vec{r} : [a,b] \to \mathbb{R}^n$ ($n=2$ or 3)

then

$$\int_C \vec{F} \cdot \underbrace{\hat{T}\, ds} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} \|\vec{r}\,'(t)\| \, dt$$

$$= \int_a^b \vec{F}(\vec{r}(t)) \cdot \underbrace{\vec{r}\,'(t)\, dt}_{d\vec{r}}$$

(Abuse of notations: $\vec{r}(t) = \vec{r}(x(t), y(t)) = x(t)\hat{i} + y(t)\hat{j}$ because of the position vector field $\vec{r} = x\hat{i} + y\hat{j}$)

$\therefore$ naturally, we denote $\boxed{d\vec{r} = \hat{T}\, ds}$

and

$$\boxed{\int_C \vec{F} \cdot \hat{T}\, ds = \int_C \vec{F} \cdot d\vec{r}}$$

eg 38: $\vec{F}(x,y,z) = z\hat{i} + xy\hat{j} - y^2\hat{k}$

$C : \vec{r}(t) = t^2\hat{i} + t\hat{j} + \sqrt{t}\,\hat{k}$, $0 \le t \le 1$

Soln $d\vec{r} = \left(2t\hat{i} + \hat{j} + \frac{1}{2\sqrt{t}}\hat{k}\right) dt$ $\qquad (\vec{r}\,'(t)\, dt)$

and $\int_C \vec{F} \cdot \hat{T}\, ds = \int_C \vec{F} \cdot d\vec{r}$

$$= \int_C \left(\sqrt{t}\,\hat{i} + (t^2 \cdot t)\hat{j} - t^2\hat{k}\right) \cdot \left(2t\hat{i} + \hat{j} + \frac{1}{2\sqrt{t}}\hat{k}\right) dt$$

$$= \int_0^1 \left(2t\sqrt{t} + t^3 - \frac{1}{2} t^{3/2}\right) dt \quad \text{(Check!)}$$

$$= \frac{17}{20} \quad \text{(check!)}$$

#

In components form:

Line integral of $\vec{F} = M\hat{i} + N\hat{j}$ along

$$C: \vec{r}(t) = g(t)\,\hat{i} + h(t)\,\hat{j}$$

can be expressed as

$$\int_C \vec{F}\cdot\hat{T}\,ds = \int_C \vec{F}\cdot d\vec{r} = \int_a^b \left(\vec{F}\cdot\frac{d\vec{r}}{dt}\right)dt$$

$$= \int_a^b (Mg' + Nh')\,dt$$

(more explicitly: $\int_a^b [M(g(t), h(t))\,g'(t) + N(g(t), h(t))\,h'(t)]\,dt$)

Note that, $\begin{cases} x = g(t) \\ y = h(t) \end{cases}$

$$\Rightarrow \begin{cases} dx = g'(t)\,dt \\ dy = h'(t)\,dt \end{cases}$$

$$\therefore \quad \boxed{\int_C \vec{F}\cdot\hat{T}\,ds = \int_C \vec{F}\cdot d\vec{r} = \int_a^b M\,dx + N\,dy}$$

Similarly, for 3-dim

$$\boxed{\int_C \vec{F}\cdot\hat{T}\,ds = \int_C \vec{F}\cdot d\vec{r} = \int_a^b M\,dx + N\,dy + L\,dz}$$

(for $\vec{F} = M\hat{i} + N\hat{j} + L\hat{k}$)

Another way to justify the notation :

$\vec{r} = (x, y, z)$ the position vector

$$\Rightarrow \quad \boxed{d\vec{r} = (dx, dy, dz)} \quad \text{(natural notation)}$$

Then $\displaystyle\int_C \vec{F}\cdot\hat{T}\,ds = \int_C \vec{F}\cdot d\vec{r} = \int_C (M, N, L)\cdot(dx, dy, dz)$

$$= \int_C M\,dx + N\,dy + L\,dz.$$

eg 39 : Evaluate $\displaystyle I = \int_C -y\,dx + z\,dy + 2x\,dz$

where $C : \vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + t\,\hat{k} \quad (0 \le t \le 2\pi)$

$$= (\cos t, \sin t, t)$$

Soln

$$I = \int_0^{2\pi} (-\sin t)\,d(\cos t) + t\,d(\sin t) + 2\cos t\,dt$$

$$= \int_0^{2\pi} (\sin^2 t + t\cos t + 2\cos t)\,dt$$

$$= \cdots = \pi \quad \text{(check!)}$$

$\divideontimes$

$\left(d\vec{r} = (-\sin t, \cos t, 1)\,dt, \text{ i.e. } \vec{r}'(t) = (-\sin t, \cos t, 1)\right)$

Physics

(1) $\vec{F}$ = Force field

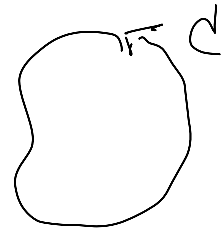
C = oriented curve

then $$\boxed{W = \int_C \vec{F}\cdot\hat{T}\,ds}$$

is the <u>workdone</u> in moving an object along C.

(2) $\vec{F}$ = velocity vector field of <u>fluid</u>

C = oriented curve



Then $$\boxed{\text{Flow} = \int_C \vec{F}\cdot\hat{T}\,ds}$$

<u>Flow</u> along the curve C.

If C is "<u>closed</u>", the flow is also called a <u>circulation</u>.

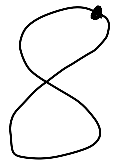
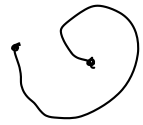
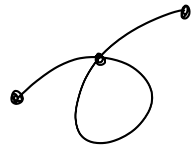
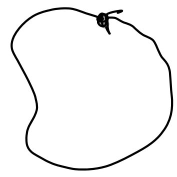
<u>Def 13</u> : A curve is said to be

(i) <u>simple</u> if it <u>does not</u> intersect with itself except possibly at end points.

(ii) <u>closed</u> if starting point = end point.

(iii) <u>simple closed curve</u> if it is both simple and closed.

Note:

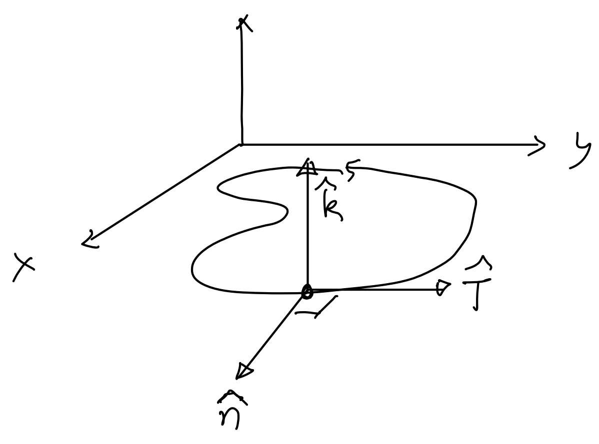
				
simple	NO	Yes	NO	Yes
closed	Yes	NO	NO	Yes

(3) $\vec{F}$ = velocity of fluid

C = oriented plane curve ($C \subset \mathbb{R}^2$) (simple, closed)

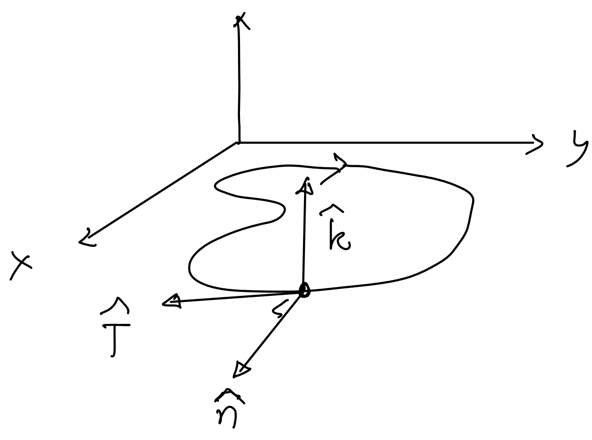
with parametrization $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$

$\hat{n}$ = outward-pointing unit normal (vector) to the curve C



$$\boxed{\hat{n} = \hat{T} \times \hat{k}}$$

if C is of anti-clockwise orientation



$$\boxed{\hat{n} = -\hat{T} \times \hat{k}}$$

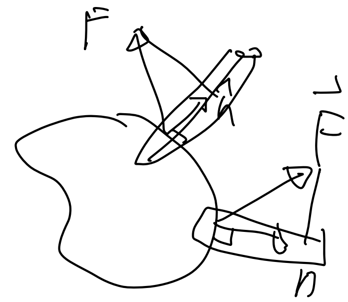
if C is of clockwise orientation.

Formula for $\hat{n}$ (wrt the parametrization $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$)

Recall $\hat{T} = \dfrac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} = \dfrac{x'(t)\hat{i} + y'(t)\hat{j}}{\|\vec{r}\,'(t)\|}$

(in arc-length parametrization: $\hat{T} = \dfrac{d\vec{r}}{ds} = \dfrac{dx}{ds}\hat{i} + \dfrac{dy}{ds}\hat{j}$)

<u>Anti-clockwise</u>:



$$\hat{n} = \hat{T} \times \hat{k} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{x'}{\|\vec{r}\,'\|} & \dfrac{y'}{\|\vec{r}\,'\|} & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$\Rightarrow \quad \hat{n} = \frac{y'(t)\hat{i} - x'(t)\hat{j}}{\|\vec{r}\,'(t)\|} \qquad \left(\text{or } \hat{n} = \frac{dy}{ds}\hat{i} - \frac{dx}{ds}\hat{j}\right)$$

<u>Clockwise</u>: $\hat{n} = \dfrac{-y'(t)\hat{i} + x'(t)\hat{j}}{\|\vec{r}\,'(t)\|}$ $\left(\text{or } \hat{n} = -\dfrac{dy}{ds}\hat{i} + \dfrac{dx}{ds}\hat{j}\right)$

$$\boxed{\underline{\text{Flux}} \text{ of } \vec{F} \text{ across } C \overset{\text{def}}{=\!=} \int_C \vec{F} \cdot \hat{n}\, ds}$$

(amount of fluid getting out of the closed curve C)

If $\vec{F} = M(x,y)\hat{i} + N(x,y)\hat{j}$

and $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$ is <u>anti-clockwise</u> parametrization of C (<u>closed curve</u>)

Then

Flux of $\vec{F}$ across C

$$= \ointctrclockwise_{C} (M\hat{i} + N\hat{j}) \cdot \left(\frac{dy}{ds}\hat{i} - \frac{dx}{ds}\hat{j}\right) ds$$

$$= \ointctrclockwise_{C} M\,dy - N\,dx$$

<u>Remark</u>:

- $\ointctrclockwise$: curve is <u>closed</u> & in <u>anti-clockwise</u> direction
- $\ointclockwise$: curve is <u>closed</u> & in <u>clockwise</u> direction

  (not a common notation)
- But in some books, only "$\oint$" is used, <u>NO</u> arrow,

  Then one needs to determine the orientation from the context.
- <u>Convention</u>: If no orientation is mentioned, "$\oint$" without arrow means <u>anti-clockwise orientation</u> (<u>positive orientation</u>)

eg 40: Let $\vec{F} = (x-y)\hat{i} + x\hat{j}$

$C: x^2+y^2 = 1$

Find the flow (anti-clockwisely) along C and flux across C.

Soln: Let $\vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j}$, $0 \le t \le 2\pi$

(Note: correct orientation) (Check!)

Then flow $= \oint_C \vec{F}\cdot\hat{T}\,ds = \oint_C \vec{F}\cdot d\vec{r}$

$$= \oint_C \left[(\cos t - \sin t)\hat{i} + \cos t\,\hat{j}\right]\cdot\left[-\sin t\,\hat{i} + \cos t\,\hat{j}\right]dt$$

$$= \int_0^{2\pi}\left[-\sin t(\cos t - \sin t) + \cos^2 t\right]dt$$

$$= \cdots = 2\pi \quad \text{(Check!)}$$

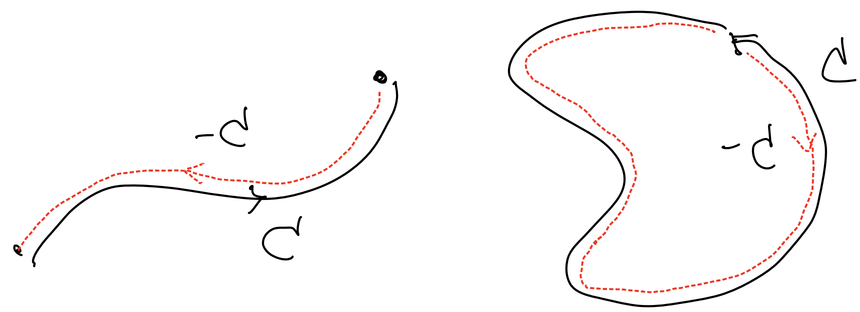
flux $= \oint_C \vec{F}\cdot\hat{n}\,ds = \oint_C M\,dy - N\,dx$ (with anti-clockwise orientation)

$$= \int_0^{2\pi}(\cos t - \sin t)\,d\sin t - \cos t\,d(\cos t)$$

$$= \int_0^{2\pi}\left[(\cos t - \sin t)\cos t + \cos t \sin t\right]dt$$

$$= \cdots = \pi \quad \text{(Check!)}$$

Remark: If C is an oriented curve, then denote by "$-C$" the oriented curve with opposite orientation



- If f is a scalar function

$$\boxed{\int_C f\,ds = \int_{-C} f\,ds}$$

as "ds" is not oriented, just "length"

- If $\vec{F}$ is a vector field

flow $$\boxed{\int_C \vec{F}\cdot\hat{T}\,ds = -\int_{-C} \vec{F}\cdot\hat{T}\,ds}$$

this $\hat{T}$ is the "$\hat{T}$ for $-C$"

More precise formula:

$$\boxed{\int_C \vec{F}\cdot\hat{T}_C\,ds = -\int_{-C} \vec{F}\cdot\hat{T}_{-C}\,ds}$$

- But for flux $$\boxed{\oint_C \vec{F}\cdot\hat{n}\,ds = \oint_{-C} \vec{F}\cdot\hat{n}\,ds}$$ $\hat{n}$ always outward

Summary:

scalar f	$\int_C f\,ds$ indep. of orientation	ds have no direction
vector $\vec{F}$ flow	$\int_C \vec{F}\cdot\hat{T}\,ds$ depends on orientation	$\hat{T}$ depends on direction
flux	$\int_C \vec{F}\cdot\hat{n}\,ds$ indep. of orientation	$\hat{n}$ always outward