eg33 : Let C be a curve in $\mathbb{R}^2$ (plane curve) (i.e. $z(t) \equiv 0$)

and it has 2 parametrizations

$$\vec{r}_1(t) = \langle \cos t, \sin t \rangle, \qquad t \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}]$$

$$\vec{r}_2(t) = \langle \sqrt{1-t^2}, -t \rangle, \qquad t \in [-1, 1]$$

Suppose $f(x,y) = x$. Find $\int_C f(x,y)\,ds$.

(We simply omit the z-variable, as C is a plane curve and f is indep. of z)

Soln: (1) $\vec{r}_1(t) = \langle \cos t, \sin t \rangle$, $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$

$$\int_C f(x,y)\,ds = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\cos t, \sin t) \| \langle \cos t, \sin t \rangle' \| \,dt$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos t \cdot 1 \,dt = 2 \quad \text{(Check!)}$$

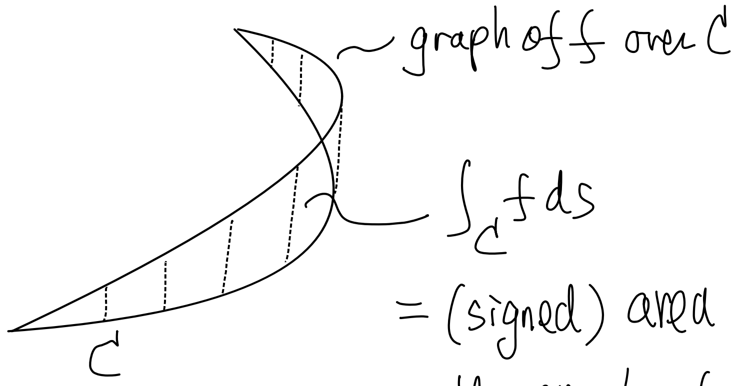
(2) $\vec{r}_2(t) = \langle \sqrt{1-t^2}, -t \rangle$, $-1 \le t \le 1$

$$\int_C f(x,y)\,ds = \int_{-1}^{1} f(\sqrt{1-t^2}, -t) \left\| \frac{d}{dt} \langle \sqrt{1-t^2}, -t \rangle \right\| dt$$

$$= \int_{-1}^{1} \sqrt{1-t^2} \cdot \sqrt{\left(\frac{d}{dt}\sqrt{1-t^2}\right)^2 + \left(\frac{d}{dt}(-t)\right)^2} \,dt$$

$$= \cdots = \int_{-1}^{1} dt = 2 \quad \text{(Check!)}$$

This verifies the fact that the line integral is indep. of the parametrization



$= $ (signed) area under the graph of f over C. ※

Prop 7: If C is a piecewise smooth curve made by joining $C_1, C_2, \cdots C_n$ <u>end-to-end</u>, then

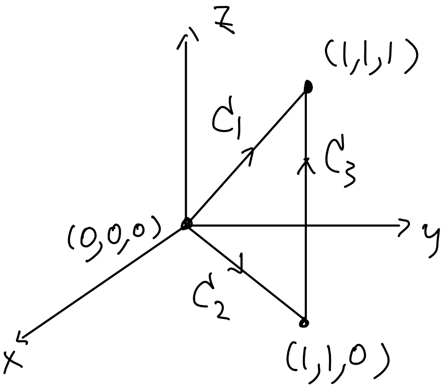
$$\int_C f\,ds = \sum_{i=1}^{n} \int_{C_i} f\,ds$$

(Pf: Clear from the remark (3) of Def 9', but C_i can be piecewise in this Prop.)

<u>Remark</u>: "end-to-end" means "end point of C_{k-1} = initial point of C_k".

eg34 : Let $f(x,y,z) = x - 3y^2 + z$ (again)

C_1, C_2, C_3 are line segments as in the figure



We already did $\int_{C_1} f\,ds = 0$ (eg32)

One can similarly calculate

$$\int_{C_2 \cup C_3} f\,ds = \int_{C_2} f\,ds + \int_{C_3} f\,ds$$

$$= -\frac{\sqrt{2}}{2} - \frac{3}{2} \qquad \text{(Ex!)}$$

(For instance, $\int_{C_3} f\,ds = \int_0^1 (1 - 3(1)^2 + t)\,dt$ (what parametrization?))

The observation is $\int_{C_1} f\,ds = 0 \neq \int_{C_2 \cup C_3} f\,ds$

even C_1 & $C_2 \cup C_3$ have the same beginning and end points!

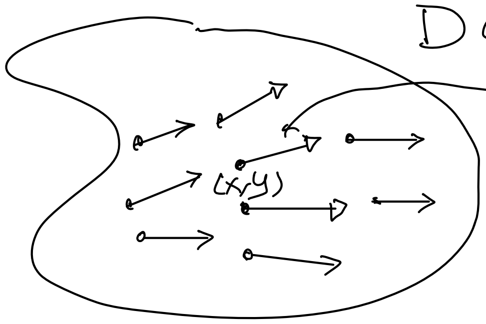
(Remark: different from 1-variable calculus)

Conclusion: Line integral of a function depends, not only on the end points, but also the path.

Vector Fields

Def 10: Let $D \subset \mathbb{R}^2$ or $\mathbb{R}^3$ be a region, then a <u>vector field</u> on D is a mapping $\vec{F}: D \to \mathbb{R}^2$ or $\mathbb{R}^3$ respectively

$D \subset \mathbb{R}^2$



$\vec{F}(x,y) \in \mathbb{R}^2$ considered as a 2-vector.

In <u>component form</u>:

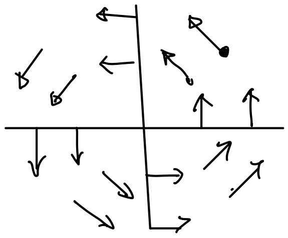
$\mathbb{R}^2$: $\vec{F}(x,y) = M(x,y)\hat{i} + N(x,y)\hat{j}$

$\mathbb{R}^3$: $\vec{F}(x,y,z) = M(x,y,z)\hat{i} + N(x,y,z)\hat{j} + L(x,y,z)\hat{k}$

where M, N, L are functions on D called the <u>components</u> of $\vec{F}$.

<u>eg 35</u> $\quad \vec{F}(x,y) = \dfrac{-y\hat{i} + x\hat{j}}{\sqrt{x^2+y^2}}$ on $\mathbb{R}^2 \setminus \{(0,0)\}$

$= -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$ (in polar coordinates)



Properties of $\vec{F}$: (i) $\|\vec{F}(x,y)\| = 1$

(ii) $\vec{F} \perp \vec{r}(x,y) = x\hat{i} + y\hat{j} = r(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$

eg36 (Gradient vector field of a function)

(i) $f(x,y) = \frac{1}{2}(x^2+y^2)$

$$\vec{\nabla} f(x,y) \stackrel{\text{def}}{=\!=} \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (x,y) = x\hat{i} + y\hat{j} = \vec{r}(x,y)$$

position vector field.

(ii) $f(x,y,z) = x$

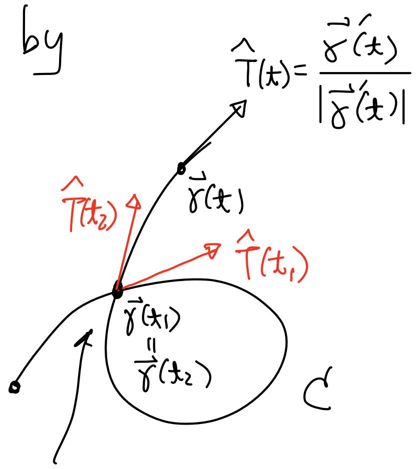
$$\vec{\nabla} f(x,y,z) \stackrel{\text{def}}{=\!=} \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \equiv (1,0,0) = \hat{i}$$

(a constant vector field)

eg37 (Vector field along a curve)

Let C be a curve in $\mathbb{R}^2$ parametrized by

$$\begin{aligned} \vec{\gamma} : [a,b] &\to \mathbb{R}^2 \\ t &\mapsto (x(t), y(t)) = \vec{\gamma}(t) \end{aligned}$$



Recall: $\hat{T} = \frac{\vec{\gamma}'(t)}{\|\vec{\gamma}'(t)\|}$

= unit tangent vector field along C (same point, but different vector)

Note: this $\hat{T}$ defined only on C (for a general curve), but not outside C.

(vector field along a curve may not come from a vector field on a region.)

Remark: for eg 37.

If we use $ds = \|\vec{r}'(t)\| dt$, then

$$\hat{T} = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = \frac{\frac{d\vec{r}}{dt}}{\frac{ds}{dt}} = \frac{d\vec{r}}{ds}$$ (by Chain rule) (if s is a function)

where "arc-length s" is defined by (up to an additive constant)

$$s(t) = \int_{t_0}^{t} \|\vec{r}'(t)\| dt,$$

A parametrization of a curve C by arc-length s is called <u>arc-length parametrization</u>:

$\vec{r}(s)$ = arc-length parametrization

$$\Rightarrow \left\| \frac{d\vec{r}}{ds}(s) \right\| = 1$$