eg31 Let $D = \{(x,y,z) \in \mathbb{R}^3 : |x|+|y|+|z| \leq 1\}$

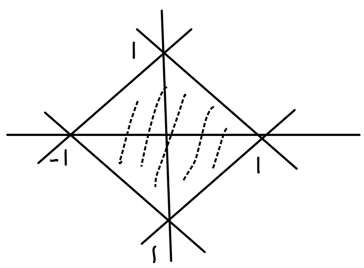
Evaluate $\iiint_D (x+y+z)^4 \, dV$.

(may use symmetry $(x,y,z) \Leftrightarrow (-x,-y,-z)$ to reduce half, but not to the 1st octant since, for instance,

$$x+y+z \longleftrightarrow x+y-z \quad \text{under } (x,y,z) \leftrightarrow (x,y,-z),$$

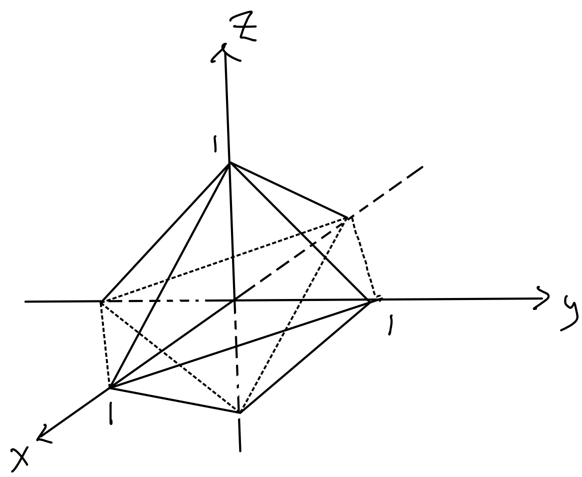
$\therefore (x+y+z)^4$ is not symmetric in all reflection with respect to the coordinates lines)

Soln If $z=0$, then $|x|+|y| \leq 1$



Boundary lines are $\begin{cases} x+y = \pm 1 \\ x-y = \pm 1 \end{cases}$

Hence



Boundary planes $\pm x \pm y \pm z = 1$

$\begin{pmatrix} + & + & + \\ - & - & - \end{pmatrix}$ let $u = x+y+z = \pm 1$

$\begin{pmatrix} + & + & - \\ - & - & + \end{pmatrix}$ let $v = x+y-z = \pm 1$

$\begin{pmatrix} + & - & - \\ - & + & + \end{pmatrix}$ let $w = x-y-z = \pm 1$

remaining pair of bdy planes $\begin{pmatrix} + & - & + \\ - & + & - \end{pmatrix}$ $u-v+w=\pm 1$

(check!)

Change of variables formula $\Rightarrow$

$$\iiint_D (x+y+z)^4\,dV = \iiint_{\substack{-1\le u,v,w\le 1\\ -1\le u-v+w\le 1}} u^4 \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dv\,dw\,du$$

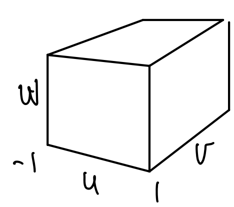
By solving
$$\begin{cases} u = x+y+z \\ v = x+y-z \\ w = x-y-z \end{cases},$$

we have
$$\begin{cases} x = \frac{1}{2}(u+w) \\ y = \frac{1}{2}(v-w) \\ z = \frac{1}{2}(u-v) \end{cases}$$ (check!)

$$\Rightarrow \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \left|\det\begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix}\right| = \left|-\frac{1}{4}\right| = \frac{1}{4}$$ (check!)

Hence
$$\iiint_D (x+y+z)^4\,dV = \iiint_{\substack{-1\le u,v,w\le 1\\ -1\le u-v+w\le 1}} \frac{u^4}{4}\,dv\,dw\,du$$
$$= A - B - C$$

where
$$A = \iiint_{-1\le u,v,w\le 1} \frac{u^4}{4}\,dv\,dw\,du$$



$$B = \iiint_{\substack{-1\le u,v,w\le 1\\ u-v+w\ge 1}} \frac{u^4}{4}\,dv\,dw\,du$$

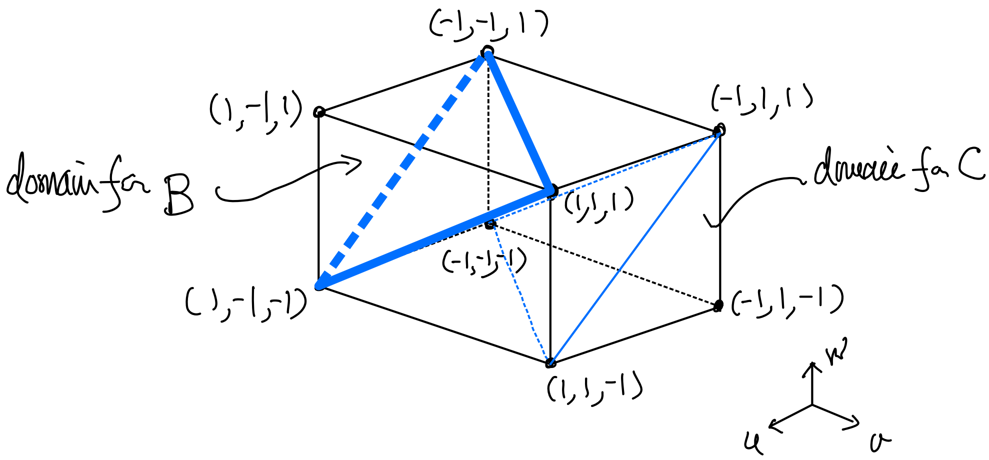
$$C = \iiint_{\substack{-1 \le u,v,w \le 1 \\ u-v+w \le -1}} \frac{u^4}{4}\, dv\,dw\,du$$

(Observation: $B = C$ by symmetry $(u,v,w) \leftrightarrow (-u,-v,-w)$)

It is clear that

$$A = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \frac{u^4}{4}\, dv\,dw\,du \;\underset{=\!=}{\text{easy}}\; \frac{2}{5} \quad \text{(check!)}$$

To handle B & C



	$u-v+w$
$(1,-1,1)$	$1-(-1)+1=3$
$(1,1,1)$	$1-1+1=1$
$(-1,-1,1)$	$(-1)-(-1)+1=1$
$(1,-1,-1)$	$1-(-1)+(-1)=1$

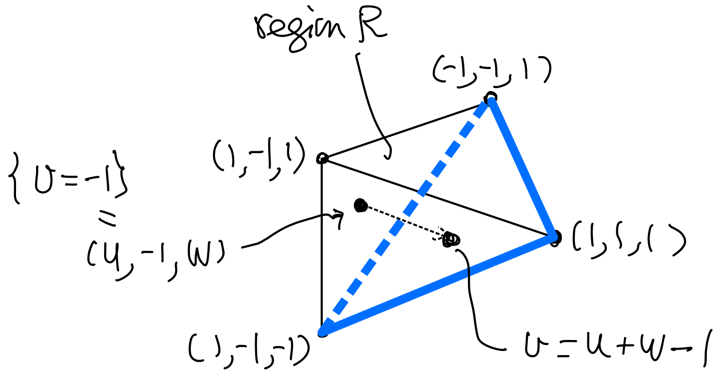
Hence the 3 points $(1,1,1)$, $(-1,-1,1)$, $(1,-1,-1)$ are on the boundary plane $u-v+w=1$

Since plane determined by 3 points, so $\{u-v+w=1\}$ is the plane passing thro. these 3 points.

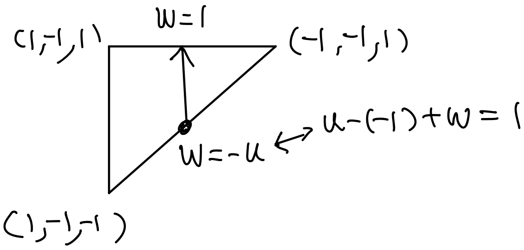
So the solid region

$$\left\{ \begin{array}{c} -1 \le u,v,w \le 1 \\ u-v+w \ge 1 \end{array} \right\}$$

for integration B is as the figure, which is of special type



$$\therefore \quad B = \iint_R \left[\int_{-1}^{u+w-1} \frac{u^4}{4}\, dv \right] dw\, du$$



$$= \int_{-1}^{1} \left[\int_{-u}^{1} \left[\int_{-1}^{u+w-1} \frac{u^4}{4}\, dv \right] dw \right] du$$

$$= \cdots = \frac{3}{35} \quad \text{(check!)}$$

Then symmetry $\Rightarrow$ $C = \frac{3}{35}$

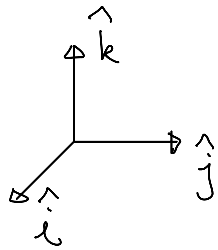
(The solid region for the integration C is determined by the 4 points $(-1,1,-1)$, $(-1,-1,-1)$, $(1,1,-1)$ & $(-1,1,1)$)

Finally $\iiint_D (x+y+z)^4\, dV = A - B - C = \frac{2}{5} - 2 \cdot \frac{3}{35} = \frac{8}{35}$ ///

Vector Analysis

Notation : Usually in textbooks, vectors are denoted by boldface $\mathbf{i}$, but hard to do it on screen, so my notation of vectors are:

$$\begin{cases} \text{general vectors} : & \vec{V}, \vec{F}, \vec{r}, \vec{\nabla}, \cdots \quad \text{(differential operator)} \\ \text{unit vectors} : & \hat{i}, \hat{j}, \hat{k}, \hat{n}, \hat{T}, \cdots \end{cases}$$



Line integrals in $\mathbb{R}^3$ ($\mathbb{R}^n$)

(path integrals)

Def 9 : The line integral of a function f on a curve (path, line) C with parametrization

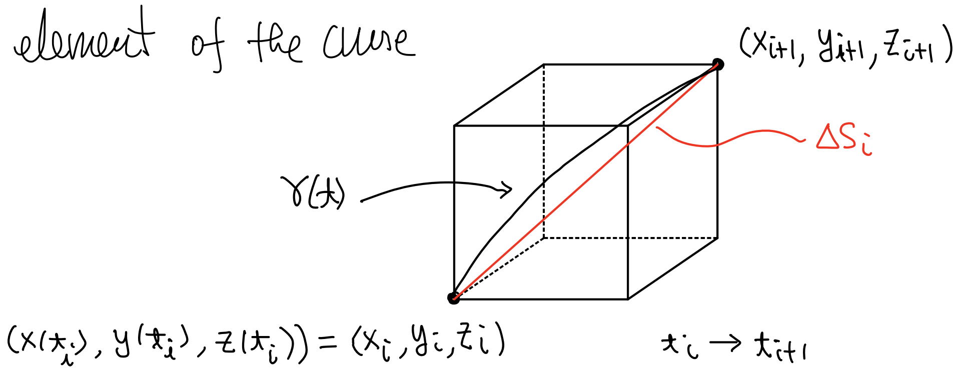
$$\begin{array}{rccc} \vec{r} : & [a,b] & \longrightarrow & \mathbb{R}^3 \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \text{(position vector)} & t & \longmapsto & (x(t), y(t), z(t)) \end{array}$$

is
$$\int_C f(\vec{r})\, ds = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(\vec{r}(t_i)) \Delta s_i$$

where P is a partition of $[a,b]$, and

$$\Delta s_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2 + (\Delta z_i)^2}$$

i.e. ds = length element of the curve



<u>Remarks</u>:

(1) If $f \equiv 1$,

$$\boxed{\int_C ds = \text{arc-length of } C}$$

(2) The definition is well-defined, i.e. the RHS in the definition is independent of the parametrization $\vec{r}(t)$.

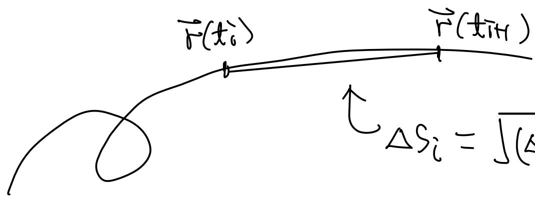
<u>Def 9'</u> (<u>Formula for line integral</u>)

Notations as in Def 9, then

$$\int_C f(\vec{r})\, ds = \int_a^b f(\vec{r}(t)) \|\vec{r}\,'(t)\|\, dt$$

where $\quad \vec{r}\,'(t) = (x'(t), y'(t), z'(t))$

Since



$$\Delta S_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2 + (\Delta z_i)^2}$$

$$= \sqrt{\left(\frac{\Delta x_i}{\Delta t_i}\right)^2 + \left(\frac{\Delta y_i}{\Delta t_i}\right)^2 + \left(\frac{\Delta z_i}{\Delta t_i}\right)^2}\ \Delta t_i$$

$$\simeq \sqrt{x'(t_i)^2 + y'(t_i)^2 + z'(t_i)^2}\ \Delta t_i$$

$$= \|\vec{r}\,'(t_i)\| \Delta t_i$$

<u>Remarks</u> (1) "$ds = \|\vec{r}'(t)\| dt$" is usually referred as the <u>arc-length element</u>,

where $\vec{r}'(t) = (x'(t), y'(t), z'(t))$ and $|\vec{r}'(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$.

(2) Suppose the curve C is parametrized by a new parameter $\tilde{t}$

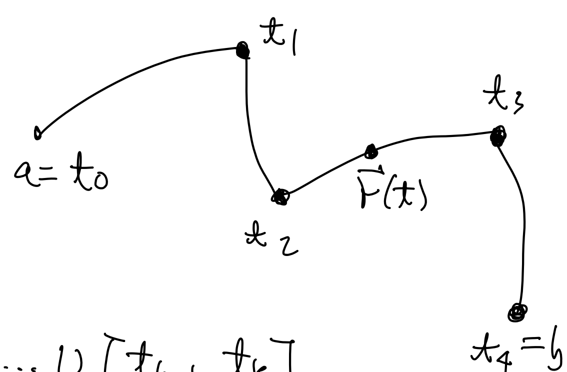
$$\begin{array}{ccc} t & \longleftrightarrow & \tilde{t} \\ \cap & & \cap \\ [a,b] & & [\tilde{a}, \tilde{b}] \end{array} \qquad \left(t \Leftrightarrow \tilde{t} \text{ is increasing } \frac{d\tilde{t}}{dt} > 0, \ \frac{dt}{d\tilde{t}} > 0 \right)$$

then $ds = \|\vec{r}'(t)\| dt = \left\| \frac{d\vec{r}}{dt}(t) \right\| dt$

$$= \left\| \frac{d\vec{r}}{d\tilde{t}} \cdot \frac{d\tilde{t}}{dt} \right\| dt = \left\| \frac{d\vec{r}}{d\tilde{t}} \right\| \left| \frac{d\tilde{t}}{dt} \right| dt = \left\| \frac{d\vec{r}}{d\tilde{t}} \right\| d\tilde{t}$$

$\therefore$ ds and hence $\int_C f(\vec{r})\, ds$ is <u>independent</u> of the parametrization of C.

(3) If $\vec{r}(t)$ is only piecewise differentiable, then the RHS of <u>Def 9'</u> becomes a sum :



If $[a,b] = [t_0, t_1] \cup \cdots \cup [t_{i-1}, t_i] \cup \cdots \cup [t_{k-1}, t_k]$, where $t_0 = a$, $t_k = b$

$\uparrow \ \uparrow$ non-differentiable point for $\vec{r}(t)$

such that $\vec{r}\big|_{[t_{i-1}, t_i]}$ is differentiable, then

$$\boxed{\int_C f(\vec{r})\,ds = \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} f(\vec{r}(t))\,|\vec{r}'(t)|\,dt}$$

eg32: $f(x,y,z) = x - 3y^2 + z$

C = line segment joining the origin and $(1,1,1)$

Find $\int_C f(x,y,z)\,ds$

Soln: Parametrize C by

$$\vec{r}(t) = t(1,1,1) = (t,t,t), \quad t \in [0,1]$$

(i.e. $x(t) = t$, $y(t) = t$, $z(t) = t$)

$\Rightarrow \vec{r}'(t) = (1,1,1), \quad \forall\, t \in [0,1]$

$\Rightarrow \|\vec{r}'(t)\| = \sqrt{3}$

Hence $\int_C f\,ds = \int_0^1 f(t,t,t)\sqrt{3}\,dt$

$= \int_0^1 (t - 3t^2 + t)\sqrt{3}\,dt = 0$ (check!)

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