Last time: we observed

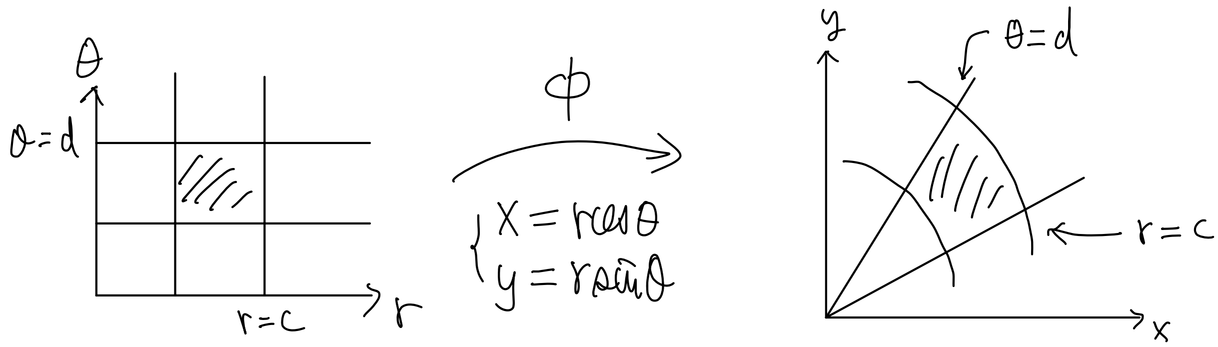
$$\int_{[a,b]} f(x)\,dx = \int_{[c,d]} f(x(u))\left|\frac{dx}{du}\right| du$$

& $\left|\frac{dx}{du}\right| \sim \frac{|\Delta x|}{|\Delta u|}$ ratio of lengths (of the coordinates)

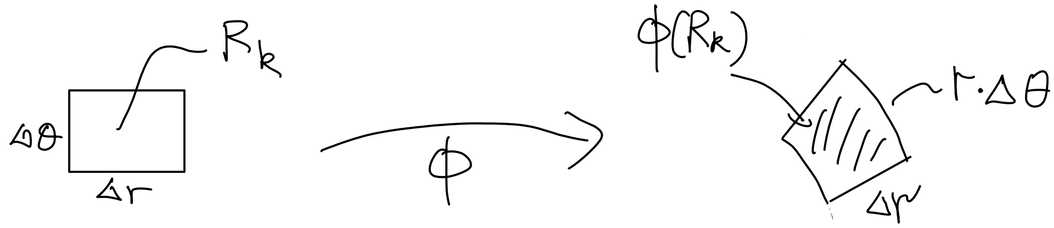
↖ 1-dim'l

Back to multiple integrals

Recall: Polar coordinates: $\iint_{(x,y)} f(x,y)\,dx\,dy = \iint_{(r,\theta)} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$



$\phi(r,\theta) = (x,y)$



Area(R_k) $\simeq \Delta r \Delta\theta$

Area($\phi(R_k)$) $\simeq r\Delta r\Delta\theta$

Hence

$$\frac{\text{Area}(\phi(R_k))}{\text{Area}(R_k)} \to r \quad \text{as "} R_k \to \text{point"}$$

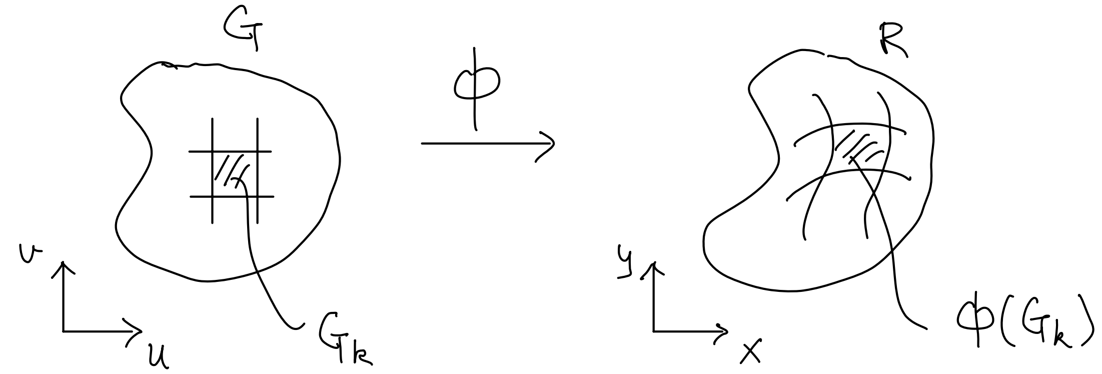
(ratio of areas, always ≥ 0)

↖ 2-dim'l

General change of coordinate formula in $\mathbb{R}^2$

Suppose $\begin{cases} x = g(u,v) \\ y = h(u,v) \end{cases}$ is denoted by $\phi(u,v) = (x,y)$,

$$\phi : \underbrace{G}_{\subset\, uv\text{-plane}} \longrightarrow \underbrace{R}_{\subset\, xy\text{-plane}}$$



Idea: We need to find

$$\frac{\text{Area}(\phi(G_k))}{\text{Area}(G_k)} \longrightarrow \, ? \quad \text{as "} G_k \to \text{point"}$$

Assume ϕ is a diffeomorphism : 1-1, onto & $\phi, \phi^{-1} \in C^1$.

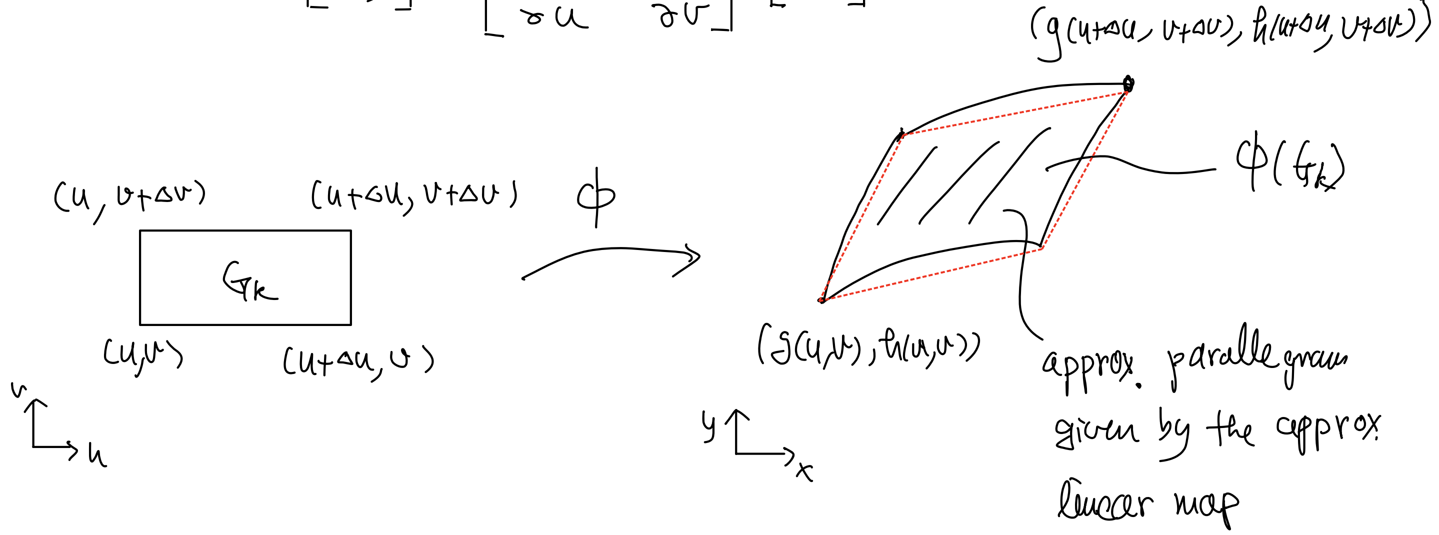
ϕ is $C^1 \Rightarrow$

$$\begin{cases} g(u+\Delta u, v+\Delta v) = g(u,v) + \dfrac{\partial g}{\partial u}\Delta u + \dfrac{\partial g}{\partial v}\Delta v + \cdots \\ h(u+\Delta u, v+\Delta v) = h(u,v) + \dfrac{\partial h}{\partial u}\Delta u + \dfrac{\partial h}{\partial v}\Delta v + \cdots \end{cases}$$

$$\Rightarrow \begin{cases} \Delta x = \Delta g = g(u+\Delta u, v+\Delta v) - g(u,v) = \dfrac{\partial g}{\partial u}\Delta u + \dfrac{\partial g}{\partial v}\Delta v + \cdots \\ \Delta y = \Delta h = h(u+\Delta u, v+\Delta v) - h(u,v) = \dfrac{\partial h}{\partial u}\Delta u + \dfrac{\partial h}{\partial v}\Delta v + \cdots \end{cases}$$

In matrix form

$$\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta v \end{bmatrix} + \cdots$$



(By linear algebra)

$$\frac{dA_{(x,y)}}{dA_{(u,v)}} \cong \frac{\text{Area}(\phi(G_k))}{\text{Area}(G_k)} \cong \left| \det \begin{bmatrix} \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} \end{bmatrix} \right|$$

$$\cong$$

$$\left(\frac{\text{“}\Delta x \Delta y\text{”}}{\text{“}\Delta u \Delta v\text{”}} \right) = \left| \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} \right|$$

<u>Def 7</u>: Define the <u>Jacobian</u> $J(u,v)$ of the "coordinates transformation" $\begin{cases} x = g(u,v) \\ y = h(u,v) \end{cases}$

by

$$J(u,v) \overset{\text{notation}}{=\!=} \frac{\partial(x,y)}{\partial(u,v)} = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$

With this notation, we should have the formula

$$\boxed{\begin{aligned}\iint_{(x,y)\in R} f(x,y)\,dx\,dy &= \iint_{(u,v)\in G} f(g(u,v),h(u,v)) \left|\det\begin{bmatrix} \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v}\end{bmatrix}\right| du\,dv \\ &= \iint_G f(x(u,v),y(u,v))\,|J(u,v)|\,du\,dv \\ &= \iint_G f(x(u,v),y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv\end{aligned}}$$

eg 28: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ $\quad \big((u,v) = (r,\theta)\big)$

$$\Rightarrow \quad J(r,\theta) = \frac{\partial(x,y)}{\partial(r,\theta)} = \det\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}\end{bmatrix} = r \quad \text{(check!)}$$

and
$$\iint_R f(x,y)\,dx\,dy = \iint_G f(r\cos\theta, r\sin\theta)\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\,d\theta$$
$$= \iint_G f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

(same formula as before.)

Thm 6: Suppose $\phi : \begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \end{pmatrix}$ is a diffeomorphism (1-1, onto, s.t. ϕ and $\phi^{-1} \in C^1$) mapping a region G (closed and bounded) in the uv-plane onto a region R (closed and bounded) in the xy-plane (except possibly on the boundary).

Suppose $f(x,y)$ is <u>continuous</u> on R, then

$$\iint_R f(x,y)\,dx\,dy = \iint_G f\circ\phi(u,v) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

<u>Notes</u>: (i) $f\circ\phi(u,v) = f(x(u,v), y(u,v))$

(ii) ϕ is a diffeomorphism $\Rightarrow \left| \dfrac{\partial(x,y)}{\partial(u,v)} \right| \neq 0$.

<u>Triple integrals</u> ("substitutions" in triple integrals)

$$\phi(u,v,w) = (x,y,z) : G^{\subset \mathbb{R}^3 \ni (u,v,w)} \longrightarrow D^{\subset \mathbb{R}^3 \ni (x,y,w)}$$

with
$$\begin{cases} x = g(u,v,w) \\ y = h(u,v,w) \\ z = k(u,v,w) \end{cases}$$
1-1, onto, cont. differentiable and <u>inverse</u> also cont. differentiable.

<u>Def 8</u> <u>Jacobian</u> (<u>determinant</u>) of transformation in $\mathbb{R}^3$

$$J(u,v,w) = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix} \left(= \det \begin{bmatrix} \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} & \frac{\partial g}{\partial w} \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} & \frac{\partial h}{\partial w} \\ \frac{\partial k}{\partial u} & \frac{\partial k}{\partial v} & \frac{\partial k}{\partial w} \end{bmatrix} \right)$$

Note: Chain rule $\Rightarrow$

$$\begin{cases} \text{2-dim} & \dfrac{\partial(x,y)}{\partial(u,v)} \cdot \dfrac{\partial(u,v)}{\partial(s,t)} = \dfrac{\partial(x,y)}{\partial(s,t)} \\[2ex] \text{3-dim} & \dfrac{\partial(x,y,z)}{\partial(u,v,w)} \cdot \dfrac{\partial(u,v,w)}{\partial(s,t,r)} = \dfrac{\partial(x,y,z)}{\partial(s,t,r)} \end{cases}$$

(Ex!)

$$\Rightarrow \begin{cases} \text{2-dim} & \dfrac{\partial(u,v)}{\partial(x,y)} = \dfrac{1}{\dfrac{\partial(x,y)}{\partial(u,v)}} \\[2ex] \text{3-dim} & \dfrac{\partial(u,v,w)}{\partial(x,y,z)} = \dfrac{1}{\dfrac{\partial(x,y,z)}{\partial(u,v,w)}} \end{cases}$$

(Ex!)

Thm 7: Under similar conditions of Thm 6

$$\iiint_D F(x,y,z)\,dx\,dy\,dz = \iiint_G F\circ\phi(u,v,w)\,|J(u,v,w)|\,du\,dv\,dw$$

$$= \iiint_G F(g(u,v,w), h(u,v,w), k(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$$

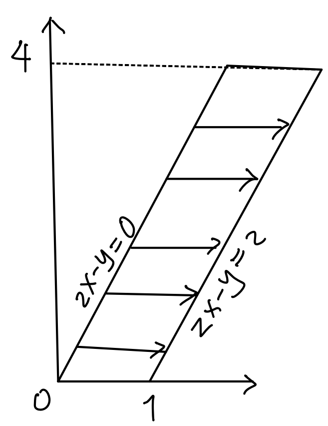
eg 29

$$\int_0^4 \int_{\frac{y}{2}}^{\frac{y}{2}+1} \frac{2x-y}{2}\,dx\,dy$$

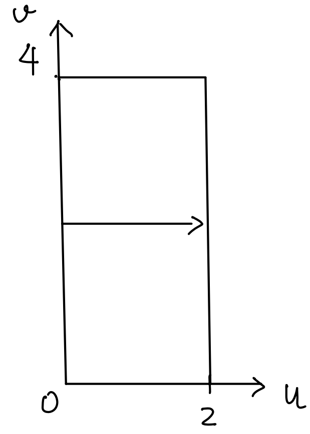
Soln

lower limit $x = \frac{y}{2} \leftrightarrow 2x - y = 0$

upper limit $x = \frac{y}{2} + 1 \leftrightarrow 2x - y = 2$



Define $\begin{cases} u = 2x-y \\ v = y \end{cases}$



Then $\begin{cases} x = \frac{1}{2}u + \frac{1}{2}v \\ y = v \end{cases}$

$\begin{cases} 2x-y=0 \iff u=0 \\ 2x-y=2 \iff u=2 \end{cases}$

$\begin{cases} y=0 \\ y=4 \end{cases} \iff \begin{cases} v=0 \\ v=4 \end{cases}$

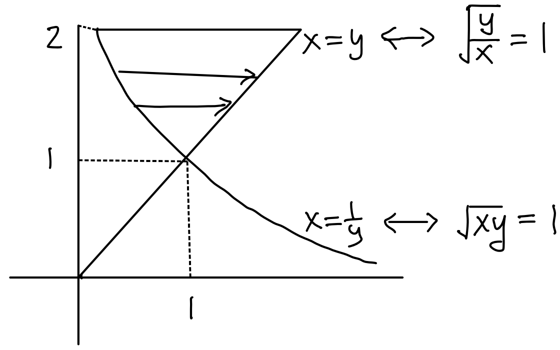
$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \det\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix} = \frac{1}{2}$$

$$\therefore \int_0^4 \int_{\frac{y}{2}}^{\frac{y}{2}+1} \frac{2x-y}{2}\,dx\,dy = \int_0^4 \int_0^2 \frac{u}{2}\left|\frac{1}{2}\right| du\,dv = 2 \quad \text{(check!)}$$

※

eg30 $$I = \int_1^2 \int_{\frac{1}{y}}^{y} \sqrt{\frac{y}{x}}\, e^{\sqrt{xy}}\, dx\,dy$$

Soln: Domain of integration

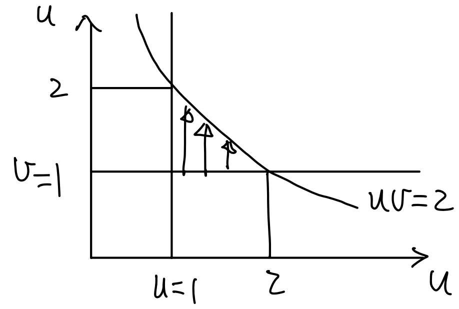


Let $\begin{cases} u = \sqrt{xy} \\ v = \sqrt{\frac{y}{x}} \end{cases}$

Express x, y in terms of u, v

$\begin{cases} x = \frac{u}{v} \\ y = uv \end{cases}$

Then
$$\begin{cases} x = y \Leftrightarrow v = 1 \\ x = \frac{1}{y} \Leftrightarrow u = 1 \\ y = 2 \Leftrightarrow uv = 2 \end{cases}$$



$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{pmatrix} = \frac{2u}{v}$$

$$I = \int_1^2 \int_{\frac{1}{y}}^{y} \sqrt{\frac{y}{x}}\, e^{\sqrt{xy}}\, dx\, dy$$

$$= \int_1^2 \int_1^{\frac{2}{u}} v e^u \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dv\, du \quad \left(= \int_1^2 \int_1^{\frac{2}{v}} v e^u \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv \right)$$

$$= \int_1^2 \int_1^{\frac{2}{u}} v e^u \frac{2u}{v}\, dv\, du$$

$$= \int_1^2 2ue^u \left(\int_1^{\frac{2}{u}} dv \right) du$$

$$= \int_1^2 2ue^u \left(\frac{2}{u} - 1 \right) du$$

$$= 2e(e-2)$$ (check!) #

eg18 (revisit) Volume of Ellipsiod

$$D = \left\{ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c} \leq 1 \right\} \quad (a, b, c > 0)$$

$$Vol(D) = 8 \int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\, dy\, dx$$

Soln Change of variables

$$\begin{cases} u = \dfrac{x}{a} \\ v = \dfrac{y}{b} \\ w = \dfrac{z}{c} \end{cases} \iff \begin{cases} x = au \\ y = bv \\ z = cw \end{cases} \Rightarrow \frac{\partial(x,y,z)}{\partial(u,v,w)} = abc$$

"New" domain (in u,v,w): $G = \{(u,v,w) : u^2+v^2+w^2 \le 1\}$

$$\begin{aligned}
\mathrm{Vol}(D) &= 8\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\,dy\,dx \\
&= 8\int_0^1 \int_0^{\sqrt{1-u^2}} \int_0^{\sqrt{1-u^2-v^2}} \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dw\,dv\,du \\
&= abc \cdot 8\int_0^1 \int_0^{\sqrt{1-u^2}} \int_0^{\sqrt{1-u^2-v^2}} dw\,dv\,du \\
&= abc\ \mathrm{Vol}(\text{Solid unit ball in } (u,v,w)\text{-space}) \\
&= abc \int_0^{2\pi}\int_0^{\pi}\int_0^1 \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta \\
&= \frac{4\pi}{3} abc
\end{aligned}$$

#