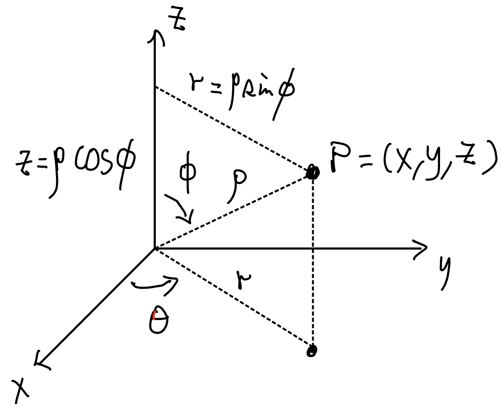
Spherical coordinates in $\mathbb{R}^3$



(ρ, ϕ, θ) where

- ρ = distance from the origin $(\rho \geq 0)$
- ϕ = angle from the <u>positive</u> z-axis to $\overline{OP}$ $\quad (0 \leq \phi \leq \pi)$
- θ = angle from cylindrical coordinate $(0 \leq \theta \leq 2\pi)$

<u>Remark</u>: If (r, θ, z) is the cylindrical coordinates of the point P, then

$$\begin{cases} r = \rho \sin\phi \\ z = \rho \cos\phi \end{cases}$$

(ρ, ϕ can be regarded as polar coordinates of the (z, r) coordinates)

In particular $\quad z^2 + r^2 = \rho^2$.

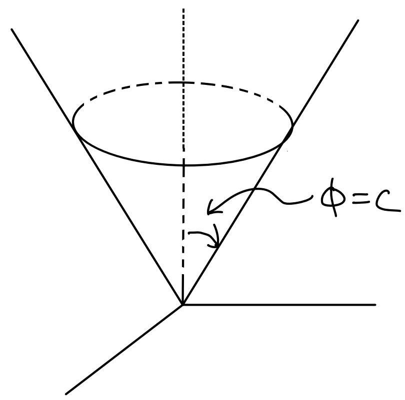
Then

$$\boxed{\begin{array}{ccccc} x & = & r\cos\theta & = & \rho\sin\phi\cos\theta \\ y & = & r\sin\theta & = & \rho\sin\phi\sin\theta \\ z & = & z & = & \rho\cos\phi \end{array}}$$

rectangular $\quad$ cylindrical $\quad$ spherical

Remark: If c is a constant, then

- $\rho = c$ $(c>0)$ describes a sphere of radius c
- $\theta = c$ describes a vertical half-plane.
- $\phi = c$ describes



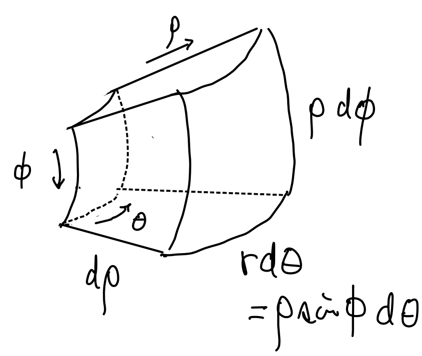
$$= \begin{cases} +\text{ve } z\text{-axis}, & \text{if } c=0 \\ -\text{ve } z\text{-axis}, & \text{if } c=\pi \\ xy\text{-plane}, & \text{if } c=\frac{\pi}{2} \\ \text{cone}, & \text{otherwise} \end{cases}$$

$\left(\begin{array}{l} \text{upward } 0<c<\frac{\pi}{2} \\ \text{downward } \frac{\pi}{2}<c<\pi \end{array}\right)$

Volume element

$$dV = dx\,dy\,dz = r\,dr\,d\theta\,dz$$

$$= (\rho\sin\phi)(\rho\,d\rho\,d\phi)\,d\theta$$



i.e. $\boxed{dV = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta}$

eg23 Convert the following into spherical coordinates

(1) $x^2+y^2+(z-1)^2=1$ (sphere)

(2) $z=-\sqrt{x^2+y^2}$ (cone)

Soln: (1) Sub.
$$\begin{cases} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi \end{cases} \quad \text{— } (*)$$

into $x^2+y^2+(z-1)^2 = 1$

$$\Leftrightarrow \quad \rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta + (\rho\cos\phi - 1)^2 = 1$$

$$\Leftrightarrow \quad \rho^2\sin^2\phi + \rho^2\cos^2\phi - 2\rho\cos\phi = 0$$

$$\Leftrightarrow \quad \rho^2 = 2\rho\cos\phi \qquad (\rho \geq 0)$$

$$\Leftrightarrow \quad \rho = 2\cos\phi .$$

(2) Sub. $(*)$ into $z = -\sqrt{x^2+y^2}$ $(= -r)$

$$\Rightarrow \quad \rho\cos\phi = -\rho\sin\phi \qquad (\rho \geq 0)$$

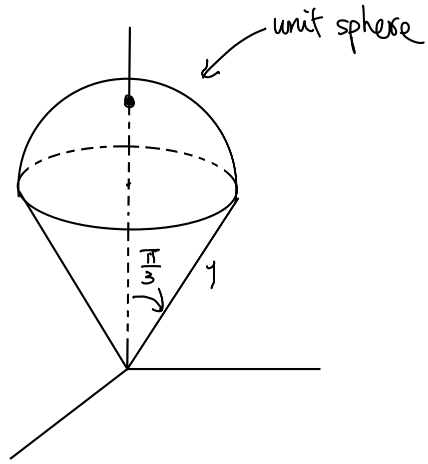
$\Rightarrow$ ($\rho = 0$ is a point $(0,0,0)$)

& $\rho \neq 0 \Rightarrow \cos\phi = -\sin\phi \qquad (0 \leq \phi \leq \pi)$

$$\Rightarrow \quad \phi = \frac{3\pi}{4}$$

eg24 (see eg22)

Volume of ice-cream cone I again, in spherical coordinates



Soln: The ice-cream cone I is given by

$$\{0 \leq \rho \leq 1,\ 0 \leq \phi \leq \frac{\pi}{3},\ 0 \leq \theta \leq 2\pi\}$$

$$\mathrm{Vol}(I) = \int_0^{2\pi}\int_0^{\frac{\pi}{3}}\int_0^1 \underbrace{\rho^2\sin\phi}\, d\rho\, d\phi\, d\theta$$

don't miss this

$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\frac{\pi}{3}}\sin\phi\, d\phi\right)\left(\int_0^1 \rho^2 d\rho\right)$$

$$= \frac{\pi}{3} \qquad \text{(Check!)}$$

<u>eg25</u>

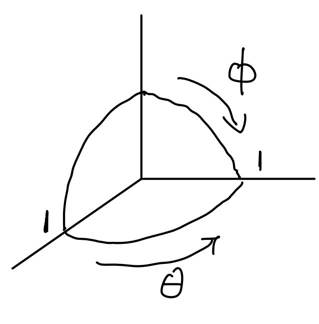
$$f(x,y,z) = \begin{cases} \dfrac{x^2+y^2}{\sqrt{x^2+y^2+z^2}}, & \text{if } (x,y,z)\neq(0,0,0) \\ 0, & \text{if } (x,y,z)=(0,0,0) \end{cases}$$

(In fact, f is continuous, but it is sufficient to know f is continuous except the origin $(0,0,0)$)

Let D = unit ball centered at origin intersecting with the 1st octant

Find the <u>average</u> of f over D.

<u>Soln</u>: D can be represented in spherical coordinates:



$$\begin{cases} 0\le\rho\le 1 \\ 0\le\phi\le\frac{\pi}{2} \\ 0\le\theta\le\frac{\pi}{2} \end{cases}$$

And $f(x,y,z) = \dfrac{x^2+y^2}{\sqrt{x^2+y^2+z^2}} = \dfrac{\rho^2\sin^2\phi}{\rho}$ $\quad$ $\left((x,y,z)\neq(0,0,0)\right)$

$= \rho\sin^2\phi$ $\quad$ ($\because f\to 0$ as $\rho\to 0 \Rightarrow f$ cts at 0)

Hence $\iiint\limits_D f(x,y,z)\,dV = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^1 \underset{\substack{\uparrow \\ \text{function}}}{(\rho \sin^2\phi)} \cdot \underbrace{\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta}_{\text{Volume element}}$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^1 \rho^3 \sin^3\phi \, d\rho \, d\phi \, d\theta$$

$$= \frac{\pi}{2} \left(\int_0^{\frac{\pi}{2}} \sin^3\phi \, d\phi \right) \left(\int_0^1 \rho^3 d\rho \right)$$

$$= \frac{\pi}{12} \quad \text{(Check!)}$$

$$\mathrm{Vol}(D) = \frac{1}{8} \mathrm{Vol}(\text{unit ball}) = \frac{1}{8} \cdot \frac{4\pi}{3} = \frac{\pi}{6}$$

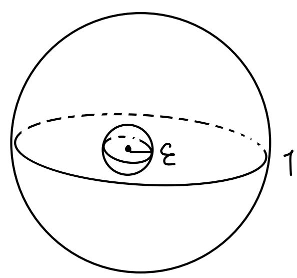
$\Rightarrow$ Average of f over $D = \dfrac{1}{\mathrm{Vol}(D)} \iiint\limits_D f(x,y,z)\,dV$

$$= \frac{1}{2} \qquad \divideontimes$$

<u>eg 26</u>: (Improper integrals)

Let $f(x,y,z) = \dfrac{1}{x^2+y^2+z^2} = \dfrac{1}{\rho^2}$ (unbounded as $\rho \to 0$)

$$g(x,y,z) = \frac{1}{(\sqrt{x^2+y^2+z^2})^3} = \frac{1}{\rho^3}$$



over unit ball $B = \{ (\rho, \theta, \phi) : 0 \le \rho \le 1 \}$

(i) Does $\lim\limits_{\varepsilon \to 0} \iiint\limits_{B \setminus B_\varepsilon} f(x,y,z)\,dV$ exist?

where $B_\varepsilon = \{ (\rho, \phi, \theta) : 0 \le \rho \le \varepsilon \}$

(ii) Does $\lim\limits_{\varepsilon \to 0} \iiint\limits_{B \setminus B_\varepsilon} g(x,y,z)\,dV$ exist?

Answer:

(i) $$\lim_{\varepsilon\to 0}\iiint_{B\setminus B_\varepsilon} f(x,y,z)\,dV = \lim_{\varepsilon\to 0}\int_0^{2\pi}\int_0^{\pi}\int_\varepsilon^1 \frac{1}{\rho^2}\cdot \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \lim_{\varepsilon\to 0} 2\pi\left(\int_0^{\pi}\sin\phi\,d\phi\right)\left(\int_\varepsilon^1 d\rho\right)$$

$$= \lim_{\varepsilon\to 0} 4\pi(1-\varepsilon) = 4\pi \quad \text{exists.}$$

(ii) $$\lim_{\varepsilon\to 0}\iiint_{B\setminus B_\varepsilon} g(x,y,z)\,dV = \lim_{\varepsilon\to 0}\int_0^{2\pi}\int_0^{\pi}\int_\varepsilon^1 \frac{1}{\rho^3}\cdot \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \lim_{\varepsilon\to 0} 2\pi\left(\int_0^{\pi}\sin\phi\,d\phi\right)\left(\int_\varepsilon^1 \frac{1}{\rho}\,d\rho\right)$$

$$= \lim_{\varepsilon\to 0} 4\pi\ln\frac{1}{\varepsilon} \quad \text{doesn't exist.}$$

Terminology:
- $f=\frac{1}{\rho^2}$ is said to be "integrable" over B (in the sense of improper integral)
- $g=\frac{1}{\rho^3}$ is said to be "non integrable" over B

[Question: determine all $\beta>0$ such that
$f=\frac{1}{\rho^\beta}$ is "integrable" over $B\subset\mathbb{R}^3$

Similar question in $\mathbb{R}^2$: determine all $\beta>0$ such that
$f=\frac{1}{r^\beta}$ is "integrable" in $\{r\le 1\}\subset\mathbb{R}^2$

(even in $\mathbb{R}^1$: $f=\frac{1}{|x|^\beta}$)]

Application of Multiple Integrals (Thomas' Calculus §15.6)

In applications, we often use the following:

In 2-dim: Let R be a region in $\mathbb{R}^2$ with density $\delta(x,y)$

- First moment about y-axis: $M_y = \iint_R x\,\delta(x,y)\,dA$
- First moment about x-axis: $M_x = \iint_R y\,\delta(x,y)\,dA$
- Mass: $M = \iint_R \delta(x,y)\,dA$
- Center of Mass (Centroid)

$$(\bar{x},\bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right)$$

In 3-dim, D solid region in $\mathbb{R}^3$ with density $\delta(x,y,z)$

- First moment:
  - about yz-plane, $M_{yz} = \iiint_D x\,\delta(x,y,z)\,dV$
  - about xz-plane, $M_{xz} = \iiint_D y\,\delta(x,y,z)\,dV$
  - about xy-plane, $M_{xy} = \iiint_D z\,\delta(x,y,z)\,dV$
- Mass: $M = \iiint_D \delta(x,y,z)\,dV$
- Center of Mass (Centroid) $(\bar{x},\bar{y},\bar{z}) = \left(\frac{M_{yz}}{M}, \frac{M_{xz}}{M}, \frac{M_{xy}}{M}\right)$

In 2-dim, R = region in $\mathbb{R}^2$ with density $\delta(x,y)$

Moments of inertia

- about x-axis : $I_x = \iint_R y^2 \delta(x,y)\, dA$
- about y-axis : $I_y = \iint_R x^2 \delta(x,y)\, dA$
- about line L : $I_L = \iint_R r(x,y)^2 \delta(x,y)\, dA$

where $r(x,y)$ = distance between (x,y) and L.

- about the origin : $I_0 = \iint_R (x^2+y^2)\delta(x,y)\, dA$

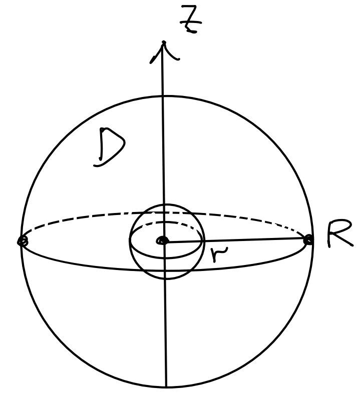
In 3-dim, D = solid region in $\mathbb{R}^3$ with density $\delta(x,y,z)$

Moments of Inertia

- around x-axis : $I_x = \iiint_D (y^2+z^2)\delta(x,y,z)\, dV$
- around y-axis : $I_y = \iiint_D (x^2+z^2)\delta(x,y,z)\, dV$
- around z-axis : $I_z = \iiint_D (x^2+y^2)\delta(x,y,z)\, dV$
- around Line L : $I_L = \iiint_D r(x,y,z)^2 \delta(x,y,z)\, dV$

where $r(x,y,z)$ = distance between (x,y,z) and L.

eg 27: Consider $D: \ r^2 \le x^2+y^2+z^2 \le R^2$
$(0<r<R)$



with density $\delta(x,y,z) \equiv \delta$
(constant density function, i.e. uniform mass)

Express I_z in term of m = Mass of D, r and R.

Soln: $I_z \overset{def}{=} \iiint_D (x^2+y^2)\,\delta(x,y,z)\,dV$

$$= \delta \iiint_D (x^2+y^2)\,dV$$

$$= \delta \int_0^{2\pi}\int_0^{\pi}\int_r^R (\rho\sin\phi)^2 \cdot \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \delta \cdot 2\pi \cdot \left(\int_0^{\pi} \sin^3\phi\, d\phi\right)\left(\int_r^R \rho^4\, d\rho\right)$$

$$= \frac{8\pi}{15}(R^5-r^5)\,\delta \qquad \text{(check!)}$$

Mass $\quad m = \iiint_D \delta(x,y,z)\,dV = \delta \iiint_D dV$

$$= \delta\, \frac{4\pi}{3}(R^3-r^3) \qquad \text{(check!)}$$

$$\Rightarrow \quad \boxed{I_z = \frac{2m}{5}\,\frac{R^5-r^5}{R^3-r^3}}$$

Observation: Two limiting cases:

(i) $r \to 0$, i.e. the whole solid ball $\boxed{I_z = \frac{2m}{5} R^2}$

(ii) $r \to R$, i.e. a (hollow) sphere made of "infinitesimally" thin sheet:

$$I_z = \lim_{r \to R} \frac{2m}{5} \cdot \frac{R^5 - r^5}{R^3 - r^3} = \frac{2m}{5} \cdot \frac{5R^4}{3R^2} \quad \text{(check!)}$$

$$\therefore \quad \boxed{I_z = \frac{2m}{3} R^2}$$

Moment of inertia of the hollow sphere

> moment of inertia of the solid ball

(assuming the same (uniform) mass m) ※

Change of Variable Formula

Review of 1-variable

In Riemann sum

$$\int_a^b f(x)\,dx = \int_{[a,b]} f(x)\,\underbrace{dx}_{(\sim |\Delta x| = \text{length of subinterval} > 0)}$$

$[a,b]$ as set (we don't care about the direction)

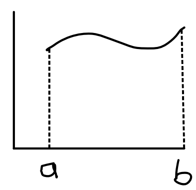
If $a > b$

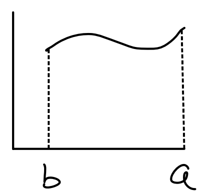
$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx = -\int_{[b,a]} f(x)\,dx$$

($[b,a]$ & $[a,b]$ represent the same set)

In summary

$$\int_a^b f(x)\,dx = \begin{cases} \displaystyle\int_{[a,b]} f(x)\,dx, & \text{if } a \le b \\ \displaystyle -\int_{[a,b]} f(x)\,dx, & \text{if } a \ge b \end{cases}$$





($[a,b] = [b,a]$ as set $= \{x : x \text{ between } a \text{ \& } b\}$)

Change of variable in 1-variable

$$\int_a^b f(x)\,dx = \int_c^d \left[f(x(u))\,\frac{dx}{du} \right] du$$

where $c = u(a)$, $d = u(b)$.

If $\frac{dx}{du} > 0$, then $d = u(b) > u(a) = c$

$$\therefore \int_a^b f(x)dx = \int_{[c,d]} \left[f(x(u))\frac{dx}{du}\right]du$$

$$= \int_{[c,d]} f(x(u))\left|\frac{dx}{du}\right|du$$

If $\frac{dx}{du} < 0$, then $d = u(b) < u(a) = c$

$$\therefore \int_a^b f(x)dx = \int_c^d \left[f(x(u))\frac{dx}{du}\right]du = -\int_{[d,c]} f(x(u))\frac{dx}{du}\,du$$

$$= \int_{[d,c]} f(x(u))\left|\frac{dx}{du}\right|du$$

Hence (in Riemann sum)

$$\frac{|\Delta x|}{|\Delta u|} \sim \left|\frac{dx}{du}\right|$$

$$\int_{[a,b]} f(x)dx = \int_{[c,d]} f(x)\left|\frac{dx}{du}\right|du$$

interpreted as a set without direction

(i.e. $\{x : x$ between c & d (inclusive) $\}$)