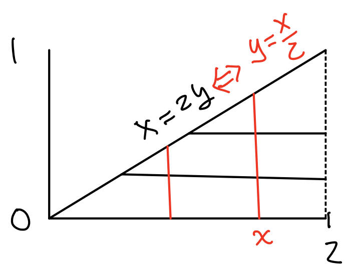
eg 20 Evaluate $\displaystyle\int_0^4\int_0^1\int_{2y}^2 \frac{4\cos(x^2)}{2\sqrt{z}}\,dx\,dy\,dz$

Soln $\displaystyle\int_0^4\int_0^1\int_{2y}^2 \frac{4\cos(x^2)}{2\sqrt{z}}\,dx\,dy\,dz$

$$= \int_0^4 \frac{2}{\sqrt{z}}\left(\int_0^1\int_{2y}^2 \cos(x^2)\,dx\,dy\right)dz$$

$$= \underbrace{\left(\int_0^1\int_{2y}^2 \cos(x^2)\,dx\,dy\right)}_{}\left(\int_0^4 \frac{2}{\sqrt{z}}\,dz\right)$$

↑ think of this as double integral over the region



By Fubini's

$$\int_0^4\int_0^1\int_{2y}^2 \frac{4\cos(x^2)}{2\sqrt{z}}\,dx\,dy\,dz = \left(\int_0^2\int_0^{\frac{x}{2}} \cos(x^2)\,dy\,dx\right)\cdot\left(\int_0^4 \frac{2}{\sqrt{z}}\,dz\right)$$

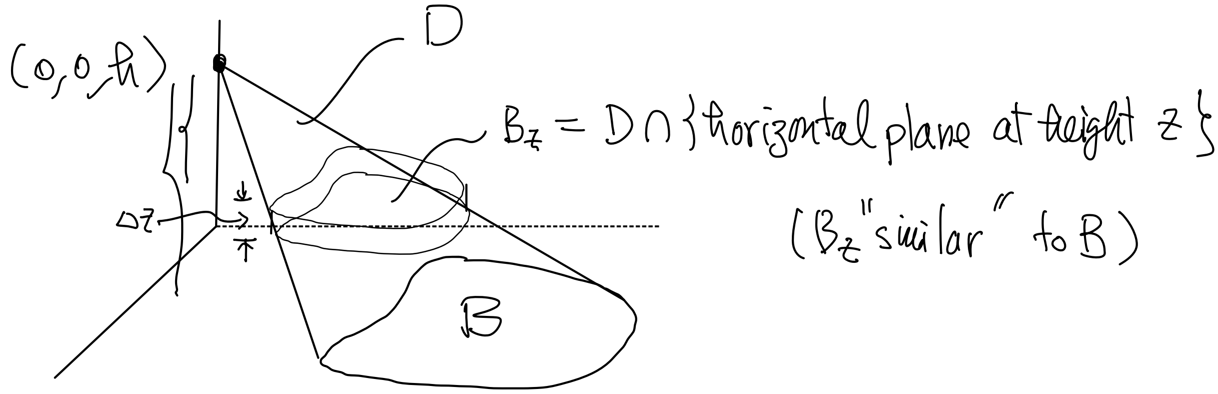
$$= \left(\int_0^2 \cos(x^2)\left(\int_0^{\frac{x}{2}} dy\right)dx\right)\cdot\left(\int_0^4 \frac{2}{\sqrt{z}}\,dz\right)$$

$$= \left(\int_0^2 \frac{x}{2}\cos(x^2)\,dx\right)\cdot\left(\int_0^4 \frac{2}{\sqrt{z}}\,dz\right)$$

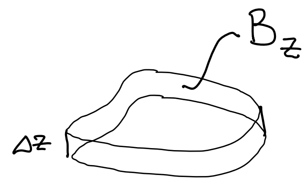
$= 2\sin 4$ (check!) (Both integrals are easy now)

eg21 Let B (base) be a "nice" subset of $\mathbb{R}^2$.

Let D = cone in $\mathbb{R}^3$ with base B on xy-plane and vertex $(0,0,h)$ $(h>0)$



How to find the volume of D?



$\text{Vol} \sim \text{Area}(B_z)\Delta z$

__Answer:__ By the concept of Riemann sum,

$$\text{Vol}(D) = \int_0^h \text{Area}(B_z)\, dz$$

ratio of heights : $\dfrac{h-z}{h} = 1 - \dfrac{z}{h}$

ratio of areas : $\dfrac{\text{Area}(B_z)}{\text{Area}(B)} = \left(1-\dfrac{z}{h}\right)^2$ by "Similarity"

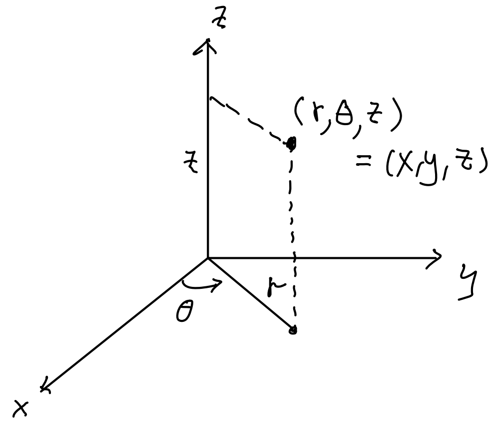
$$\Rightarrow \quad \text{Vol}(D) = \int_0^h \left(1-\frac{z}{h}\right)^2 \text{Area}(B)\, dz$$

$$= \text{Area}(B) \int_0^h \left(1-\frac{z}{h}\right)^2 dz$$

$$= \frac{h}{3}\,\text{Area}(B)$$ (Check!)

※

Cylindrical Coordinates in $\mathbb{R}^3$



- (r,θ) = polar coordinates for the xy-plane ($r \geq 0$)

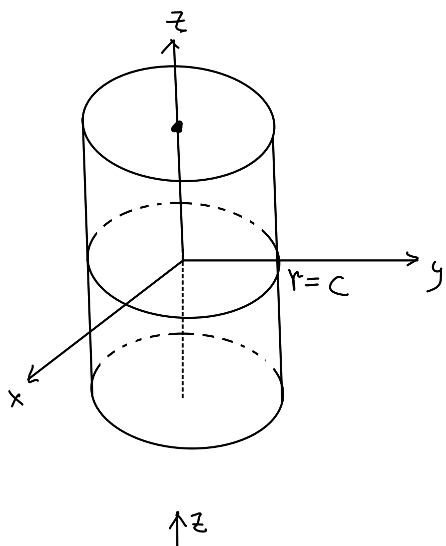
- z = rectangular vertical coordinate

Then a point $P = (x,y,z)$ can be represented by (r,θ,z), where

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

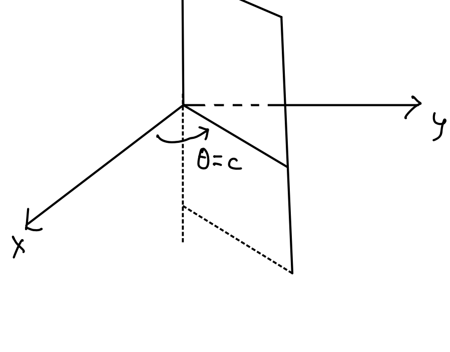
And (r,θ,z) is called the <u>cylindrical coordinates</u> for $\mathbb{R}^3$.

<u>Remark 1</u>: (Let c be a constant)



- $r = c$ $(c > 0)$ describes a <u>cylinder</u>

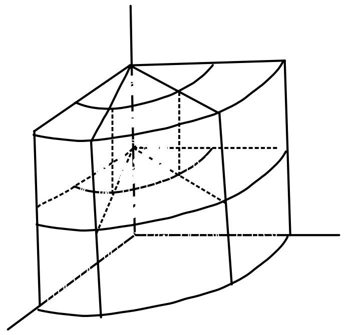
- $\theta = c$ $(0 \leq c \leq 2\pi)$ describes a vertical half-plane



- $z = c$ describes a horizontal plane (as in rectangular coordinates)

Remark 2 : We can define cylindrical coordinates in other directions.

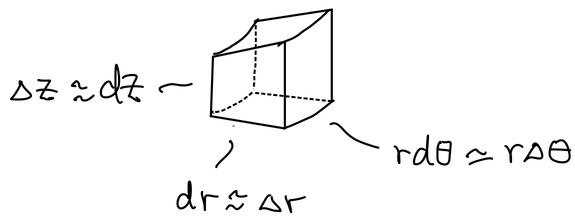
e.g. $\begin{cases} x = x \\ y = r\cos\theta \\ z = r\sin\theta \end{cases}$ (Ex: draw the cylinder $r = c$)



Volume element

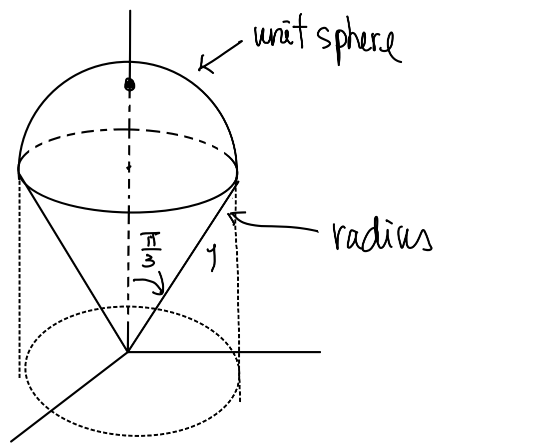
$$dV = \underbrace{dx\,dy}_{\updownarrow}\,\underbrace{dz}_{\updownarrow}$$

$$= r\,dr\,d\theta \cdot dz$$

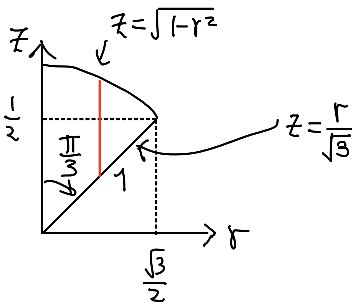


(order of the integration can be changed)

eg22 (see also eg24)
Find the volume of the Ice-cream cone I given as in the figure.



Soln: θ fixed



$\left(\begin{array}{l} z = \sqrt{1-r^2} = f_2(r,\theta) \\ \quad\quad\quad \text{indep. of } \theta \\ z = \frac{r}{\sqrt{3}} = f_1(r,\theta) \end{array} \right)$

Fubini's $\Rightarrow$ $\displaystyle Vol(D) = \int_0^{2\pi}\left(\int_0^{\frac{\sqrt{3}}{2}}\left(\int_{\frac{r}{\sqrt{3}}}^{\sqrt{1-r^2}} r\,dz\,dr\right)d\theta\right.$

$$= 2\pi \int_0^{\frac{\sqrt{3}}{2}} \left(\sqrt{1-r^2} - \frac{r}{\sqrt{3}}\right) r\,dr$$

$$= \cdots = \frac{\pi}{3} \quad \text{(Check!)}$$

$\#$