(Cont'd)

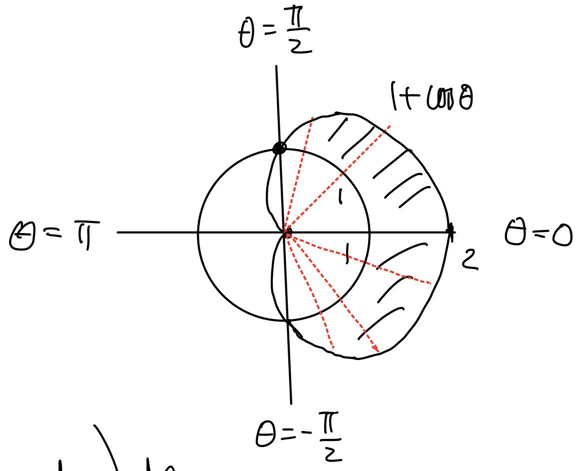
Soln Intersections:

$$1 = r = 1+\cos\theta$$

$$\Leftrightarrow \quad \cos\theta = 0$$

$$\Leftrightarrow \quad \theta = \frac{\pi}{2} + k\pi \quad (k\in\mathbb{Z})$$

Choose $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$



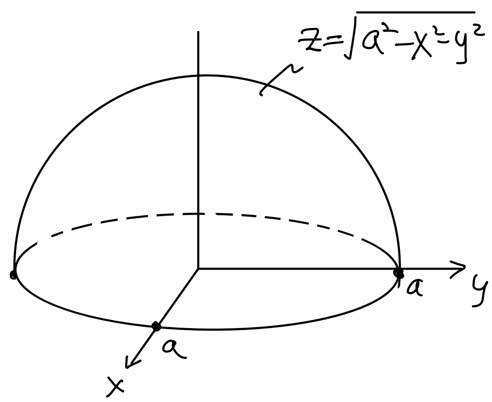
$$\Rightarrow \iint_R f(x,y)\,dA = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\int_1^{1+\cos\theta} \frac{1}{r}\cdot r\,dr\right)d\theta$$

$$= \cdots = 2 \quad \text{(check!)}$$

eg16: Let $z=\sqrt{a^2-x^2-y^2}$ be a function defined on

$$R=\{(x,y): x^2+y^2\le a^2\}$$

The graph of z is the (upper) hemisphere of radius a. Find the average height of the hemisphere.



Soln: Average height $= \dfrac{1}{\text{Area}(R)}\iint_R z\,dA$

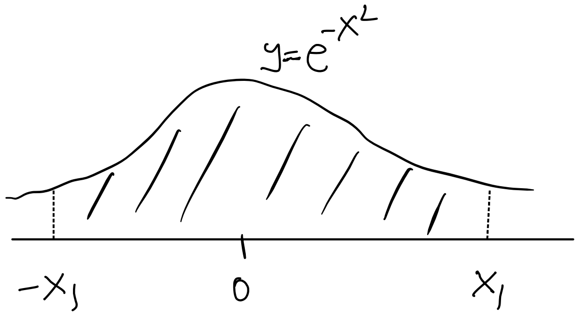
$$= \frac{1}{\pi a^2}\int_0^{2\pi}\left(\int_0^a \sqrt{a^2-r^2}\cdot r\,dr\right)d\theta$$

$$= \frac{2a}{3} \quad \text{(check!)}$$

#

eg17 (Improper integral)

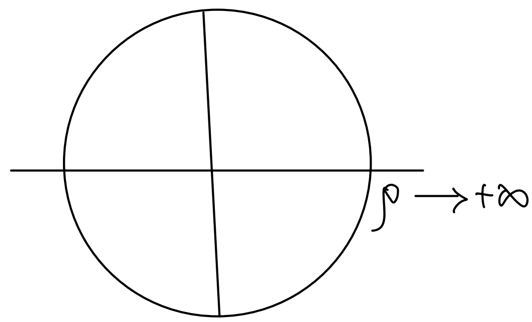
Find $\int_{-\infty}^{\infty} e^{-x^2}\,dx$



Soln:

Consider $\iint_{\mathbb{R}^2} e^{-x^2-y^2}\,dA$ (Also an improper integral)

$$= \lim_{\rho\to+\infty} \iint_{\{x^2+y^2\le\rho^2\}} e^{-(x^2+y^2)}\,dA$$

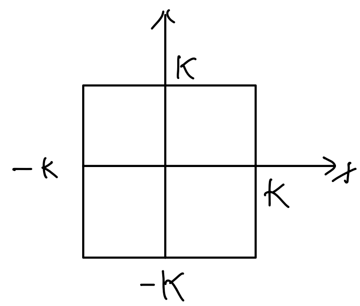


$$= \lim_{\rho\to+\infty} \int_0^{2\pi}\left(\int_0^{\rho} e^{-r^2}\, r\,dr\right)d\theta$$

$$= \lim_{\rho\to+\infty} \pi\left(1-e^{-\rho^2}\right) \quad \text{(Check!)}$$

$$= \pi$$

On the other hand



$$\iint_{\mathbb{R}^2} e^{-x^2-y^2}\,dA$$

$$= \lim_{K\to+\infty} \int_{-K}^{K}\left(\int_{-K}^{K} e^{-x^2-y^2}\,dx\right)dy$$

$$= \lim_{K\to+\infty} \left(\int_{-K}^{K} e^{-x^2}dx\right)\left(\int_{-K}^{K} e^{-y^2}dy\right) \quad \text{(check!)}$$

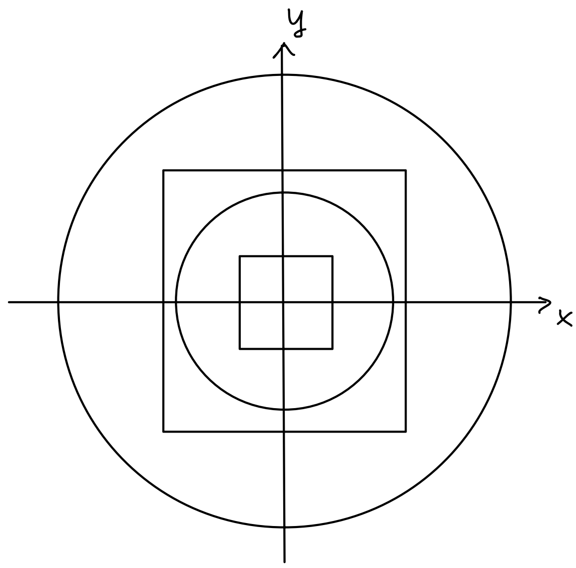
$$= \left(\lim_{K\to+\infty}\int_{-K}^{K} e^{-x^2}dx\right)^2$$

$$= \left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)^2 \quad \Rightarrow \quad \int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$$

※

Caution: we are calculating $\iint_{\mathbb{R}^2} e^{-x^2-y^2}\,dA$ in <u>two</u> different limiting processes. Why are they equal?

<u>Hints</u>: $e^{-x^2} > 0$ and



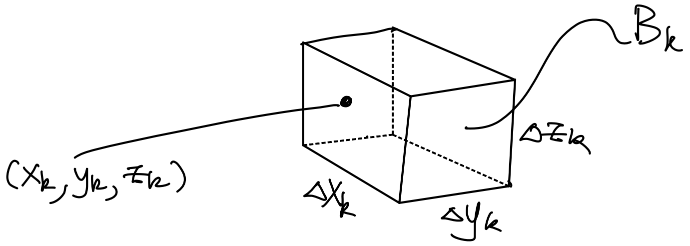
Triple Integrals

> **Def 5** Let $f(x,y,z)$ be a function defined on a (closed and bounded) rectangular box
>
> $$B = [a,b] \times [c,d] \times [r,s]$$
>
> Then the triple integral of f over the box B is
>
> $$\iiint_B f(x,y,z)\,dV = \lim_{\|P\| \to 0} \sum_k f(x_k, y_k, z_k)\,\Delta V_k$$
>
> if it exists.

where (i) $P = P_1 \times P_2 \times P_3$ is a subdivision of B into sub-rectangular boxes by partitions P_1, P_2 & P_3 of $[a,b]$, $[c,d]$, and $[r,s]$ respectively. And

$$\|P\| = \max(\|P_1\|, \|P_2\|, \|P_3\|)$$

(ii) (x_k, y_k, z_k) is an arbitrary point in a sub-rectangular box B_k



(iii) $\Delta V_k = \mathrm{Vol}(B_k) = \Delta x_k \Delta y_k \Delta z_k$.

Thm 4 (Fubini's Theorem for Triple Integrals (1^{st} form))

If $f(x,y,z)$ is continuous (in fact, "absolutely" integrable is sufficient) on $B=[a,b]\times[c,d]\times[r,s]$, then

$$\iiint_B f(x,y,z)\,dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz$$

Note: Interchanging the order of the coordinates, we also have

$$\iiint_B f(x,y,z)\,dV = \int_r^s \int_a^b \int_c^d f(x,y,z)\,dy\,dx\,dz$$

$$= \cdots \text{ in any order of } dx,\ dy,\ dz.$$

Def 6 (Triple integral over a general region $D\subset\mathbb{R}^3$)

Let $f(x,y,z)$ be a function on a closed and bounded region $D\subset\mathbb{R}^3$. Then

$$\iiint_D f(x,y,z)\,dV \overset{\text{def}}{=\!=} \iiint_B F(x,y,z)\,dV$$

where B is a closed and bounded rectangular box containing D, and

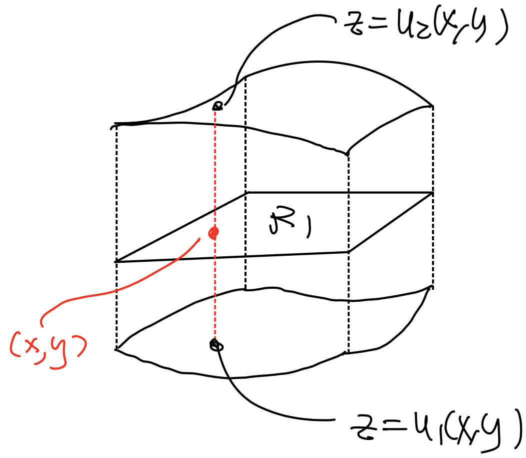
$$F(x,y,z)=\begin{cases} f(x,y,z), & \text{if } (x,y,z)\in D \\ 0, & \text{if } (x,y,z)\in B\setminus D. \end{cases}$$

Note: As in double integral, this definition is well-defined.

Special types of closed and bounded region $D \subset \mathbb{R}^3$

(1) $D = \{(x,y,z) : (x,y) \in R_1,\ u_1(x,y) \le z \le u_2(x,y)\}$

$(u_1(x,y) \le u_2(x,y),\ u_1 \not\equiv u_2)$



(2) $D = \left\{ (x,y,z) : (x,z) \in R_2,\ v_1(x,z) \le y \le v_2(x,z) \right\}$

$(v_1 \le v_2,\ v_1 \not\equiv v_2)$

(3) $D = \{(x,y,z) : (y,z) \in R_3,\ w_1(y,z) \le x \le w_2(y,z)\}$

$(w_1 \le w_2,\ w_1 \not\equiv w_2)$

where R_i, $i=1,2,3$ are <u>closed and bounded</u> plane regions and u_1, u_2; v_1, v_2; w_1, w_2 are continuous wrt the corresponding variables.

<u>Thm 5</u> (<u>Fubini's Thm for Triple integrals (Strong form)</u>)

Let $f(x,y,z)$ be a <u>continuous</u> (<u>absolutely integrable</u>) function on D. If D is of type (1) as above, then

$$\iiint_D f(x,y,z)\,dV = \iint_{R_1} \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \right] dx\,dy$$

Similarly for types (2) and (3).

Note: Particularly, we have (using Fubini's for double integrals)

if $D = \left\{ (x,y,z) : \begin{array}{l} a \le x \le b, \ g_1(x) \le y \le g_2(x) \\ u_1(x,y) \le z \le u_2(x,y) \end{array} \right\}$

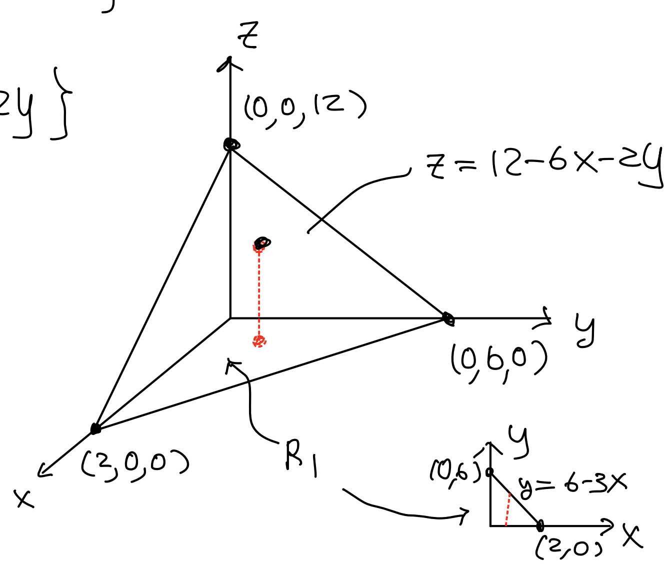
(i.e. R_1 is of type (1) as in double integrals), then

$$\boxed{\iiint_D f(x,y,z)\,dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dy\,dx}$$

Similarly for other types.

Prop6: The propositions 1–4 for double integrals also hold for triple integrals over closed and bounded region in $\mathbb{R}^3$.

eg17: Volume of the bounded region D in the 1st octant enclosed by the plane $6x+2y+z=12$



soln: $D = \{(x,y)\in R_1, \ 0 \le z \le 12-6x-2y\}$

$= \left\{ \begin{array}{l} 0 \le x \le 2, \ 0 \le y \le 6-3x, \\ 0 \le z \le 12-6x-2y \end{array} \right\}$

$\Rightarrow Vol(D) = \iiint_D 1\,dV$

$$= \int_0^2 \int_0^{6-3x} \int_0^{12-6x-2y} 1 \cdot dz\, dy\, dx$$

$$= \cdots = 24 \quad (\text{Check!})$$

<u>Remark</u>: For D of type 1,

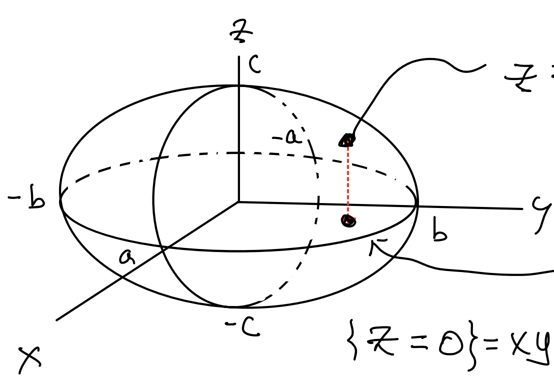
$$\mathrm{Vol}(D) = \iiint_D 1\, dV \overset{\text{Fubini}}{=\!=} \iint_{R_1} \left[\int_{u_1(x,y)}^{u_2(x,y)} 1\, dz \right] dA$$

$$= \iint_{R_1} [u_2(x,y) - u_1(x,y)]\, dA$$

Formula for volume between two graphs $z = u_2(x,y)$ and $z = u_1(x,y)$.

<u>eg18</u>: <u>Volume of Ellipsoid</u>

$$D = \left\{ (x,y,z) : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1 \right\} \quad (a,b,c > 0)$$



$$z = c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$$

$$y = b\sqrt{1 - \frac{x^2}{a^2}}$$

$\{z = 0\} = xy\text{-plane}$

Soln By symmetry, we can consider the 1st octant only,

and $Vol(D) = 8 \cdot$ volume of D in the 1st octant

$$= 8 \cdot \int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \left[\int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} 1 \, dz \right] dy\, dx$$

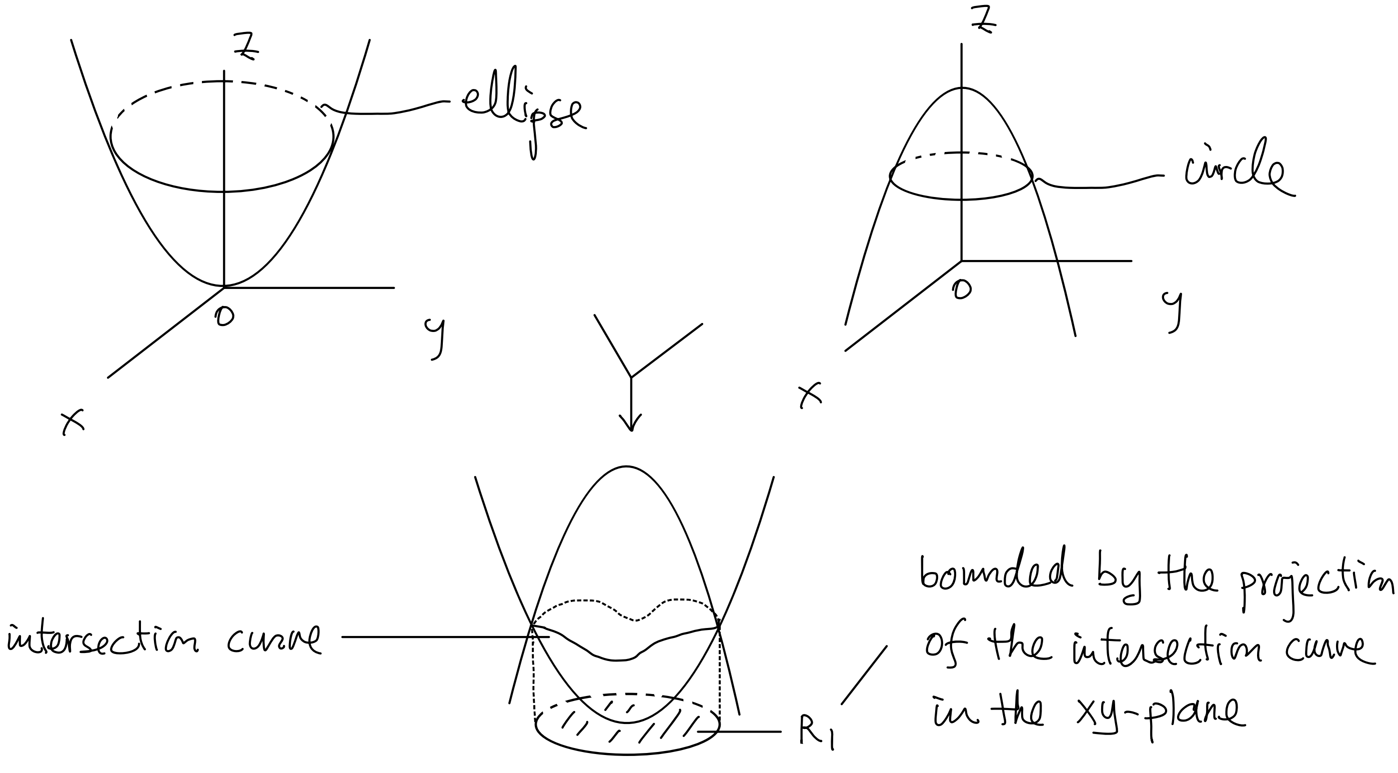
$$= 8 \int_0^a \left(\int_0^{b\sqrt{1-\frac{x^2}{a^2}}} c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}} \, dy \right) dx$$

$$= \cdots = \frac{4\pi abc}{3} \quad \text{(optional exercise)}$$

[In fact, we will have a better way to calculate this volume by "change of variables formula" (later)]

eg19: Find the volume of D enclosed by

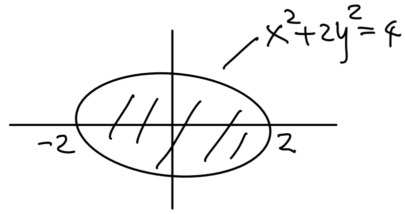
$$z = x^2 + 3y^2 \quad \text{and} \quad z = 8 - x^2 - y^2$$



Soln: Intersection curve: $x^2+3y^2 = z = 8-x^2-y^2$

$\Rightarrow$ $x^2+2y^2=4$ is the projection (in xy-plane) of the intersection curve (a ellipse)

So R_1 is



$$\Rightarrow D = \left\{ (x,y) \in R_1 = \{x^2+2y^2 \leq 4\},\ x^2+3y^2 \leq z \leq 8-x^2-y^2 \right\}$$

$$= \left\{ -2 \leq x \leq 2,\ -\sqrt{\frac{4-x^2}{2}} \leq y \leq +\sqrt{\frac{4-x^2}{2}},\ x^2+3y^2 \leq z \leq 8-x^2-y^2 \right\}$$

Fubini $\Rightarrow$

$$\mathrm{Vol}(D) = \int_{-2}^{2} \int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} \int_{x^2+3y^2}^{8-x^2-y^2} 1\, dz\, dy\, dx$$

$$= \int_{-2}^{2} \frac{4\sqrt{2}}{3} (4-x^2)^{3/2} dx \qquad \text{(check!)}$$

$$= 8\pi\sqrt{2} \qquad \text{(check!)}$$

[For those interested in the intersection (space) curve (in parametric form)

$x = 2\cos t,\ y = \sqrt{2}\sin t,\ z = 4+2\sin^2 t$

$(0 \leq t \leq 2\pi)$]