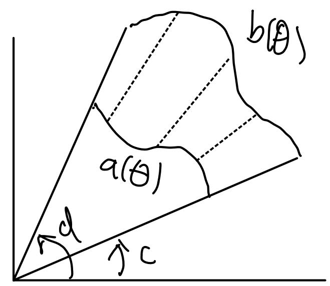
More generally

Thm3: If R is a (closed and bounded) region with

$$a(\theta) \leq r \leq b(\theta) \text{ and } c \leq \theta \leq d$$

$$(0 \leq a(\theta) \leq b(\theta),\ a(\theta) \not\equiv b(\theta))$$



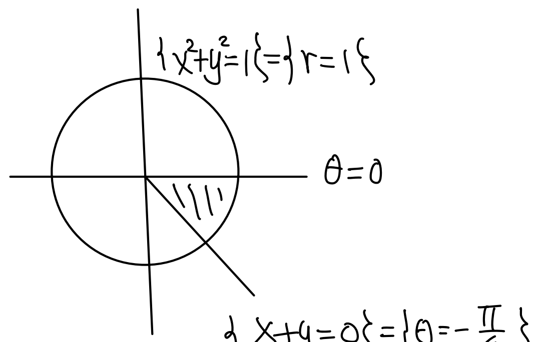
And $f: R \to \mathbb{R}$, then

$$\iint_R f(x,y)\,dA = \int_c^d \left(\int_{a(\theta)}^{b(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr \right) d\theta$$

(remember this extra "r")

eg12: Back to our previous example 9

$$f(x,y) = x = r\cos\theta$$



$$\int_{-\frac{1}{\sqrt{2}}}^{0} \int_{-y}^{\sqrt{1-y^2}} x\,dx\,dy$$

$$= \int_{-\frac{\pi}{4}}^{0} \left(\int_0^1 r\cos\theta\, r\,dr \right) d\theta$$

$$= \int_{-\frac{\pi}{4}}^{0} \left(\cos\theta \int_0^1 r^2\,dr \right) d\theta$$

$$= \cdots = \frac{1}{3\sqrt{2}} \qquad \text{(check!)}$$

Much easier than before!

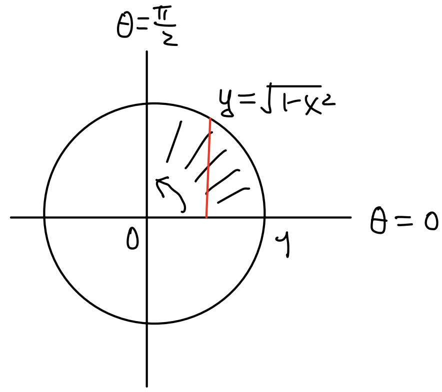
eg 13 Convert integrals between Cartesian and Polar coordinates

(a) $\int_0^{\frac{\pi}{2}}\int_0^1 r^3 \sin\theta\cos\theta \, dr\, d\theta$

(b) $\int_1^2\int_0^{\sqrt{2x-x^2}} y\, dy\, dx$

Soln: (a) $\int_0^{\frac{\pi}{2}}\int_0^1 r^3 \sin\theta\cos\theta\, dr\, d\theta$

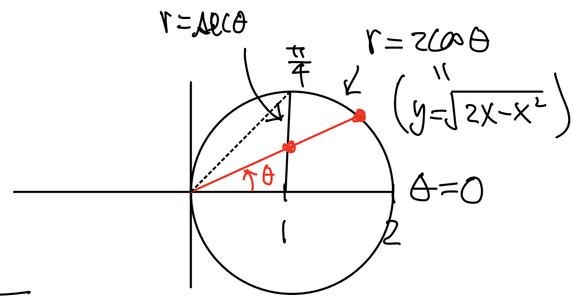
$= \int_0^{\frac{\pi}{2}}\left(\int_0^1 (r\cos\theta)(r\sin\theta)\, r\, dr\right) d\theta$



$= \int_0^1\left(\int_0^{\sqrt{1-x^2}} x y \, dy\right) dx \qquad \left(\text{or} = \int_0^1\left(\int_0^{\sqrt{1-y^2}} x y\, dx\right) dy\right)$

(b) $\int_1^2\int_0^{\sqrt{2x-x^2}} y\, dy\, dx$

$= \int_1^2\left[\int_0^{\sqrt{2x-x^2}} y\, dy\right] dx$



$\Rightarrow$ The region is $1\le x\le 2,\ 0\le y\le\sqrt{2x-x^2}$

The curve $x=1 \Leftrightarrow r\cos\theta = 1 \Leftrightarrow r=\sec\theta$

" " $y=\sqrt{2x-x^2} \Leftrightarrow r\sin\theta = \sqrt{2r\cos\theta - r^2\cos^2\theta}$

$\vdots$

$\Leftrightarrow \quad r^2 = 2r\cos\theta$

$\Leftrightarrow \quad r = 2\cos\theta \qquad$ (for $r>0$)

Hence $\int_1^2\int_0^{\sqrt{2x-x^2}} y\, dy\, dx = \int_0^{\frac{\pi}{4}}\left(\int_{\sec\theta}^{2\cos\theta} r^2\sin\theta\, dr\right) d\theta$.

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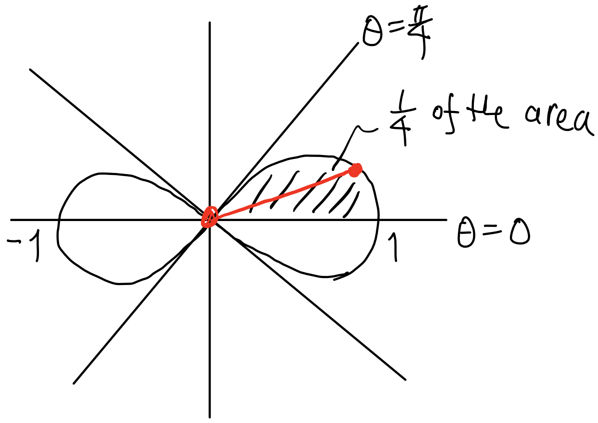
eg 14: Find area enclosed by $r^2 = 4\cos 2\theta$.

Remark: r is "not really" a function of θ, it should be regarded as a "level set":



(i) there is no solution when

$\frac{\pi}{4} < \theta < \frac{3\pi}{4}$ & $\frac{5\pi}{4} < \theta < \frac{7\pi}{4}$

(ii) in term of (x,y) coordinates

$$F(x,y) = (x^2+y^2)^2 - 4(x^2-y^2) = 0 \quad \text{(check!)}$$

which has a critical point at $(x,y)=(0,0)$ $(\vec{\nabla}F(0,0)=\vec{0})$ on the level set. (One cannot apply "Implicit Function Theorem" at the critical point $(0,0)$)

(later for more detail)

Soln: By the symmetry

$$\text{Area} = 4\int_0^{\frac{\pi}{4}} \left(\int_0^{\sqrt{4\cos 2\theta}} 1 \cdot r\,dr \right) d\theta$$

$$= 8\int_0^{\frac{\pi}{4}} \cos 2\theta \, d\theta = 4 \quad \text{(check)}$$

※

eg 15: Integrate $f(x,y) = \frac{1}{\sqrt{x^2+y^2}}$ over the region R bounded between

$$\begin{cases} r = 1+\cos\theta & \text{(cardiod)} \\ r = 1 \end{cases}$$

and outside the circle $r=1$.

Soln (next time)

Region: $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$,

$1 \le r \le 1+\cos\theta$

⋮

Ans $= 2$ (check!)