Furthermore, we have

Prop 3: Let $R = [a,b] \times [c,d]$ be a closed rectangle, $f(x,y)$ and $g(x,y)$ be functions on R, and $k \in \mathbb{R}$ is a constant.

(1) If f & g are integrable over R, then $f \pm g$ and kf are integrable over R.

(2) In the case of (1), we have

$$\iint_R [f \pm g](x,y)\,dA = \iint_R f(x,y)\,dA \pm \iint_R g(x,y)\,dA$$

and

$$\iint_R kf(x,y)\,dA = k \iint_R f(x,y)\,dA.$$

Pf: Omitted (Obvious from the concept of Riemann sum.)

Remark: This Prop 3 implies that the set of integrable functions over (fixed) R forms a "vector space over $\mathbb{R}$", "(double) integral" is linear.

Prop 4: (a) If $f(x,y) \geq 0$ is an integrable function on a closed rectangle R, then

$$\iint_R f(x,y)\,dA \geq 0 .$$

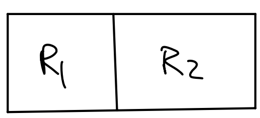
(b) If R_1 and R_2 be two closed rectangles such that $\text{int}\, R_1 \cap \text{int}\, R_2 = \phi$, then

$$\iint_{R_1 \cup R_2} f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$$

for integrable function over $R_1 \cup R_2$.

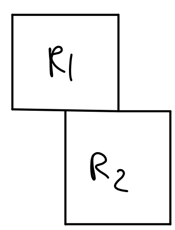
Pf: Omitted (Obvious from the concept of Riemann sum)

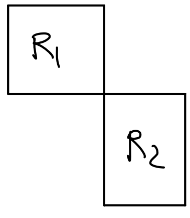
Note: Various situations for $\text{int}\, R_1 \cap \text{int}\, R_2 = \phi$

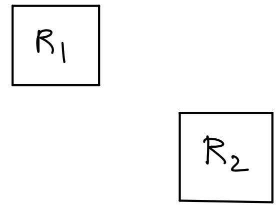
(1) 

$R_1 \cap R_2 =$ common edge

$\text{int}\, R_1 \cap \text{int}\, R_2 = \phi$

(2) 

(3) 

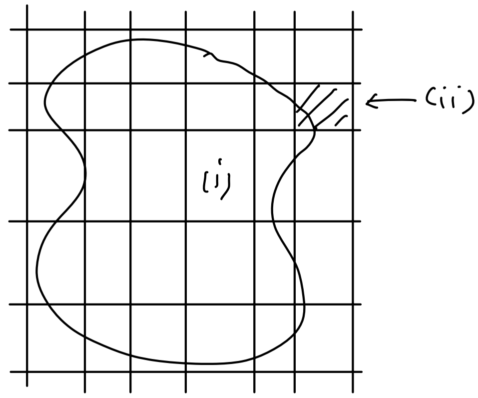
(4) 

We haven't define $\iint_{R_1 \cup R_2} f(x,y)\,dA$ for cases (2) − (4)!

Hence we need to define double integrals over general regions.

Double Integrals over General Regions

For non-rectangular bounded (closed) region R, one can define similarly the concept of "Riemann sum".

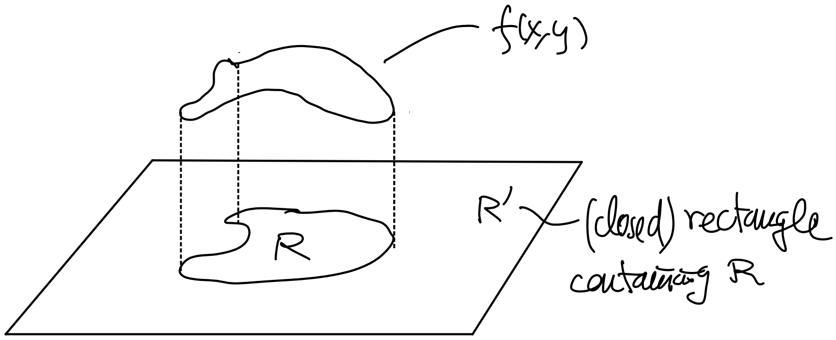


There are two ways to form the "sum"

(i) sum over all subrectangles completely inside R

(ii) sum over all subrectangles with non-empty intersection with R.

Or, one can define "the integrals" as follows



Def 2: Let R be a bounded region and $f(x,y)$ be a function defined on R. For any rectangle $R' \supset R$, define

$$F(x,y) = \begin{cases} f(x,y), & (x,y) \in R \\ 0, & (x,y) \in R' \setminus R \end{cases}$$

Then the integral of f over R is defined by

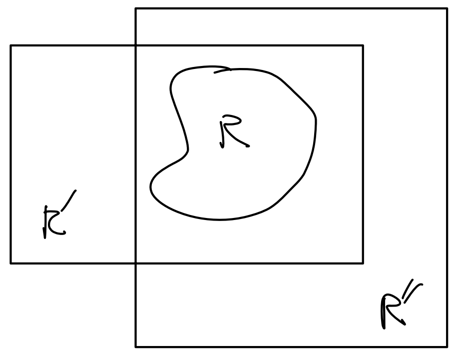
$$\iint_R f(x,y)\,dA = \iint_{R'} F(x,y)\,dA$$

Remark: The definition is well-defined (i.e. doesn't depend on the choice of R'): If R'' is another rectangle s.t. $R'' \supset R$ and

$$\tilde{F}(x,y) = \begin{cases} f(x,y), & (x,y) \in R \\ 0, & (x,y) \in R'' \setminus R \end{cases}$$

Then $\displaystyle\iint_{R''} \tilde{F}(x,y)\, dA = \iint_{R'} F(x,y)\, dA$

(by Prop 4 (b))



Prop 5: The propositions 1–4 hold if we replace "closed rectangle" by "closed and bounded region"

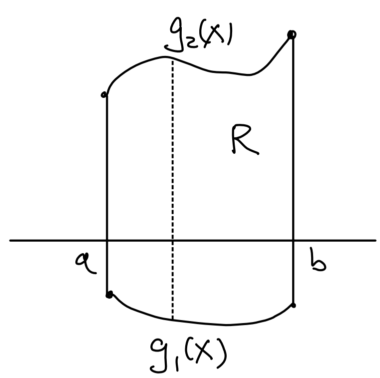
(together with the Prop 2')

Important special types of bounded regions R

Type (1): $R = \{(x,y) : a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$

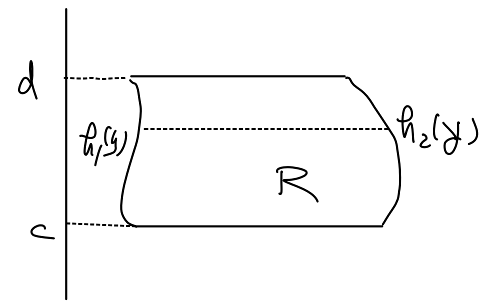
where g_1 and g_2 are "continuous" functions on $[a,b]$.

($g_1 \le g_2$, but $g_1 \not\equiv g_2$)



Type (2) : $R = \{(x,y) : h_1(y) \le x \le h_2(y),\ c \le y \le d\}$

where h_1 and h_2 are "continuous" functions on $[c,d]$
($h_1 \le h_2$, but $h_1 \not\equiv h_2$)



For these 2 types of bounded regions, we have

<u>Thm 2</u> (<u>Fubini's Thm (Stronger version)</u>)
Let $f(x,y)$ be a <u>continuous</u> function on a <u>closed and bounded</u> region R.

(1) If R is of type (1) as above, then

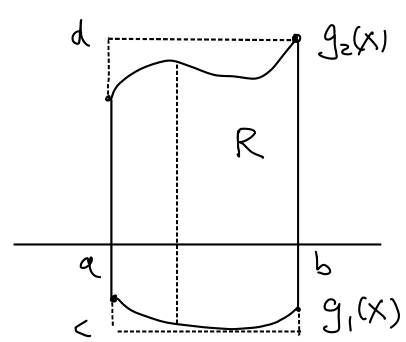
$$\iint_R f(x,y)\,dA = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy \right) dx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

(2) If R is of type (2) as above, then

$$\iint_R f(x,y)\,dA = \int_c^d \left(\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx \right) dy = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

<u>Pf</u>: Type (1) : Extend $f(x,y)$ to $F(x,y)$ as in the definition on the rectangle $R' = [a,b] \times [c,d]$ such that

$$c = \min_{[a,b]} g_1(x), \quad d = \max_{[a,b]} g_2(x)$$



By definition 2, $\iint_R f(x,y)\,dA = \iint_{R'} F(x,y)\,dA$

$$= \int_a^b \left(\int_c^d F(x,y)\,dy \right) dx \qquad \left(\text{Fubini (1}^{\text{st}}\text{ form)}\right)$$

[f continuous on $R \Rightarrow F$ continuous on R' except possibly on the boundary (curve(s)) of R. Hence by Prop 2', F (in fact $|F|$) is integrable over R'. And the Fubini theorem (1st form) is in fact true for "absolutely" integrable functions on a rectangle.]

Now $F(x,y) = 0$ for $y < g_1(x)$ and $y > g_2(x)$,

and $F(x,y) = f(x,y)$ for $g_1(x) \le y \le g_2(x)$.

$$\therefore \iint_R f(x,y)\,dA = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy \right) dx.$$

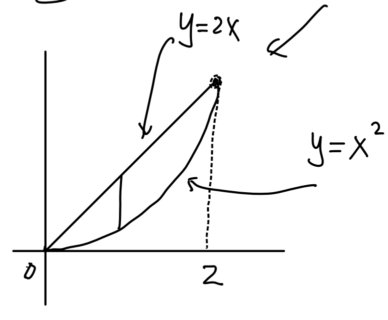
Type (2) can be proved similarly. ※

<u>eg 7</u> Integrate $f(x,y) = 4y+2$

over the region bounded by $y = x^2$ and $y = 2x$. Calculate!

Soln: By Fubini's,

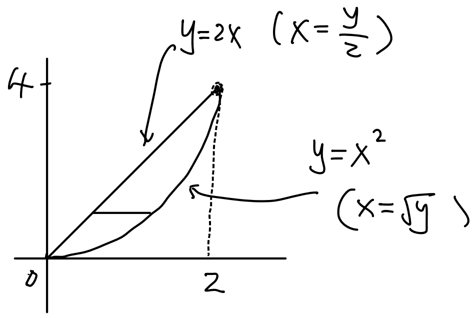
$$\iint_R f(x,y)\,dA = \int_0^2 \int_{x^2}^{2x} (4y+2)\,dy\,dx$$



$$= \int_0^2 (-2x^4+6x^2+4x)\,dx \qquad \text{(Check!)}$$

$$= \frac{56}{5} \quad \text{(check!)}$$

Or, using the fact that R is also of type (2):



$$\iint_R f(x,y)\,dA = \int_0^4 \int_{\frac{y}{2}}^{\sqrt{y}} (4y+2)\,dx\,dy$$

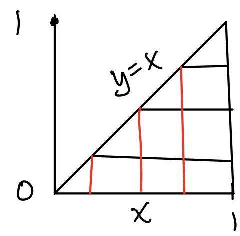
$$= \int_0^4 (4y+2)\left(\sqrt{y} - \frac{y}{2}\right) dy$$

$$= \cdots = \frac{56}{5} \quad \text{(check!)}$$

<u>eg8</u>: Evaluate $\displaystyle\int_0^1 \int_y^1 \frac{\sin x}{x}\,dx\,dy$.

<u>Soln</u>: Regard $\displaystyle\int_0^1 \int_y^1 \frac{\sin x}{x}\,dx\,dy$ as a double integral of $\frac{\sin x}{x}$ over the region

$$\begin{cases} y \le x \le 1 \\ 0 \le y \le 1 \end{cases}$$



By Fubini's

$$\int_0^1 \int_y^1 \frac{\sin x}{x}\,dx\,dy = \int_0^1 \int_0^x \frac{\sin x}{x}\,dy\,dx$$

$$= \int_0^1 \sin x\,dx = 1 - \cos 1.$$

(Caution: $f(x,y) = \frac{\sin x}{x}$ doesn't define at $x=0$. Why can we use Fubini? ($f(x,y) \ge 0$ & cts except on a line!))

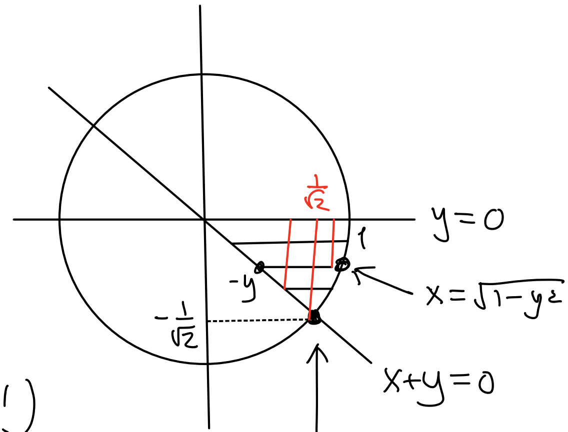
eg 9: Find $\iint_R x\,dA$, where R is the region in the right half-plane bounded by $y=0$, $x+y=0$, and the unit circle.

Soln: By Fubini's

$$\iint_R x\,dA = \int_{-\frac{1}{\sqrt{2}}}^{0}\left(\int_{-y}^{\sqrt{1-y^2}} x\,dx\right)dy$$

$$= \int_{-\frac{1}{\sqrt{2}}}^{0}\left(\tfrac{1}{2}-y^2\right)dy \qquad \text{(Check!)}$$

$$= \frac{1}{3\sqrt{2}} \qquad \text{(check!)}$$



need to solve eqts $\begin{cases} x^2+y^2=1 \\ x+y=0 \end{cases}$ for the point & get $y=-\frac{1}{\sqrt{2}}$ (reject $y=+\frac{1}{\sqrt{2}}$)

Alternatively,

$$\iint_R x\,dA = \int_0^{\frac{1}{\sqrt{2}}}\left(\int_{-x}^{0} x\,dy\right)dx + \int_{\frac{1}{\sqrt{2}}}^{1}\left(\int_{-\sqrt{1-x^2}}^{0} x\,dy\right)dx$$

$$= \int_0^{\frac{1}{\sqrt{2}}} x^2\,dx + \int_{\frac{1}{\sqrt{2}}}^{1} x\sqrt{1-x^2}\,dx \qquad \text{(Check!)}$$

$$= \frac{1}{3\sqrt{2}} \qquad \text{(Check!)}$$

Applications

(1) Area (of "good" region $R \subset \mathbb{R}^2$)

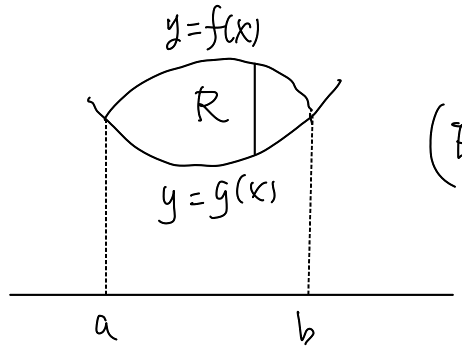
$$\boxed{\text{Def 3}: \quad \text{Area}(R) = \iint_R 1 \, dA}$$

Then Fubini's Thm implies the well-known formula

$$\text{Area}(R) = \int_a^b [f(x) - g(x)] \, dx$$

if R is the region bounded by the curves

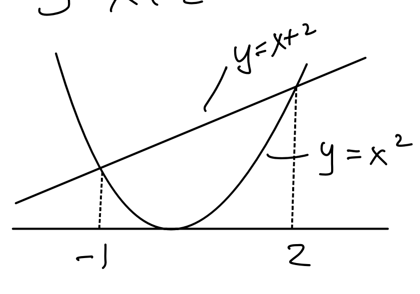
$y = f(x)$ and $y = g(x)$ for $a \le x \le b$ (with $\begin{cases} f(a) = g(a), \\ f(b) = g(b), \\ g(x) \le f(x) \end{cases}$)



(Ex!)

eg 10 Area bounded by $y = x^2$ and $y = x+2$

Soln: Solve $\begin{cases} y = x^2 \\ y = x+2 \end{cases} \Rightarrow x = -1, 2$



Then Fubini's

$$\text{Area} = \int_{-1}^{2} (x+2-x^2) \, dx = \frac{9}{2} \quad (\text{check!})$$

(2) Average (of a function over a region)

Let $f : R^{\subset \mathbb{R}^2} \to \mathbb{R}$ be an integrable function

Def 4: The average value of f over R $$= \frac{1}{\text{Area}(R)} \iint_R f(x,y)\, dA$$

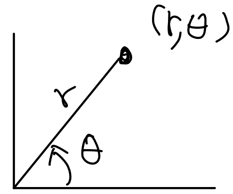
eg 11 Let $f(x,y) = x\cos(xy)$, $R = [0,\pi] \times [0,1]$

Find average of f over R.

Soln:

$$\text{Average of } f \text{ over } R = \frac{1}{\text{Area}(R)} \iint_R f(x,y)\, dA$$
$$= \frac{1}{\pi} \int_0^{\pi} \int_0^1 x\cos(xy)\, dy\, dx$$
$$= \frac{1}{\pi} \int_0^{\pi} \sin x\, dy = \frac{2}{\pi} \quad \text{(check!)}$$

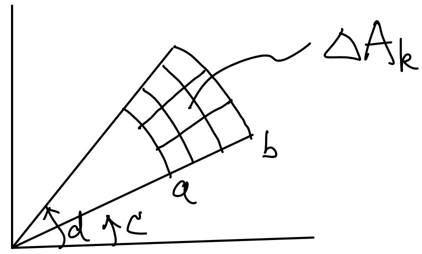
Double integral in polar coordinates



$(r,\theta) \leftrightarrow (x,y)$

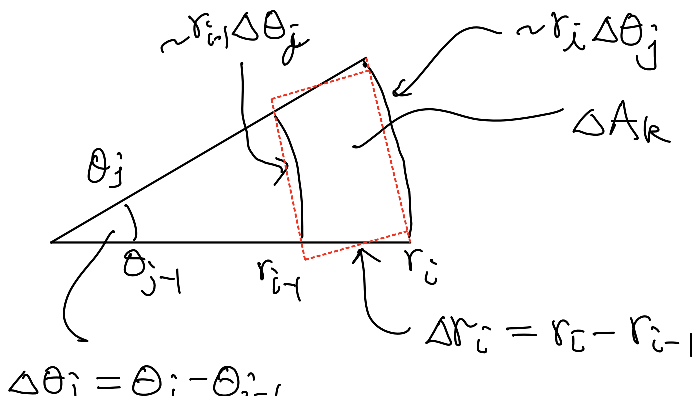
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$\begin{cases} a \le r \le b \\ c \le \theta \le d \end{cases}$$



Idea: $\sum_k f(\text{point}_k)\,\Delta A_k$

What is ΔA_k (approximately)?



$\therefore \quad \Delta A_k \simeq (r_i \Delta\theta_j)\cdot \Delta r_i \quad \left(\simeq (r_{i-1}\Delta\theta_j)\cdot\Delta r_i\right)$

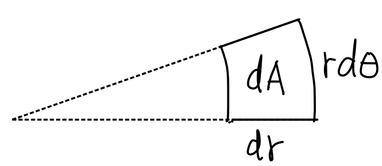
Hence $\quad \Delta A_k \cong \Delta x\,\Delta y \simeq (r\Delta\theta)\cdot\Delta r$

So $\quad \displaystyle\iint_R f(x,y)\,dA = \iint_R f(x,y)\,\underbrace{dx\,dy}$

$$= \iint_R f(r\cos\theta, r\sin\theta)\,\underbrace{r\,dr\,d\theta}$$

Method to remember the formula

$dA = dx\,dy = r\,dr\,d\theta$



Double integral of f over $R = \{(r,\theta): a \le r \le b,\ c \le \theta \le d\}$ in polar coordinates is

$$\iint_R f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta = \int_c^d \left(\int_a^b f(r,\theta)\, r\,dr \right) d\theta$$

$$= \int_a^b \left(\int_c^d f(r,\theta)\, d\theta \right) r\,dr$$

where $f(r,\theta)$ is the simplified notation for $f(r\cos\theta, r\sin\theta)$

Remark: This is a special case of the change of variables formula.

The "extra" factor "r" in the integrand is in fact

$$r = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$ the Jacobian determinant of the change of variables.