eg4 : Some times the "order" of the iterated integrals is important in practice :

Find $\iint_{[0,1]\times[0,\pi]} x\sin(xy)\,dA$.

Soln : $\iint_{[0,1]\times[0,\pi]} x\sin(xy)\,dA = \int_0^{\pi}\left[\int_0^1 x\sin(xy)\,dx\right]dy$

$= \int_0^{\pi}\left(-\frac{\cos y}{y} + \frac{\sin y}{y^2}\right)dy$ (integration-by-parts)

Not easy to integrate !

On the other hand,

$\iint_{[0,1]\times[0,\pi]} x\sin(xy)\,dA = \int_0^{1}\left[\int_0^{\pi} x\sin(xy)\,dy\right]dx$

$= \int_0^1 (-\cos\pi x + 1)\,dx$

$= 1$ (easy !) ※

Caution : Not all functions are integrable over a (closed) rectangle.

Remark : • To show "integrable", needs to show that for all partitions and for all points (x_k, y_k) in the subrectangles, the Riemann sum

$S(f, P) \to$ the same number (as $\|P\| \to 0$)

- To <u>disprove</u> "integrable", needs to find, for examples

(i) <u>some</u> P with <u>some</u> choose of (x_k, y_k) such that

$$\lim_{\|P\| \to 0} S(f, P) \text{ doesn't exist.}$$

(ii) <u>some</u> P with <u>different</u> (x_k, y_k) & (x_k', y_k') such that

$$\underset{\text{with } (x_k, y_k)}{S(f, P)} \to a \neq b \leftarrow \underset{\text{with } (x_k', y_k')}{S'(f, P)}$$

<u>eg 5</u>: Let $R = [0,1] \times [0,1]$

$$f(x,y) = \begin{cases} 0, & \text{if both } x \text{ and } y \text{ are rational} \\ 1, & \text{otherwise.} \end{cases}$$

Then f is <u>not</u> integrable over R. (using (ii))

<u>Soln</u>:

$\forall$ partition P of $R = R_1 \cup \cdots \cup R_n$ (R_k subrectangles)
One can find points $(x_k, y_k) \in R_k$, for any k, such that both x_k, y_k are rational (why?)

The corresponding Riemann sum equals

$$S_n(f, P) = \sum_{k=1}^{n} f(x_k, y_k) \Delta A_k = \sum_{k=1}^{n} 0 \cdot \Delta A_k = 0$$

$$\to 0, \text{ as } \|P\| \to 0$$

On the other hand, we can also find $(x_k', y_k') \in R_k$ such that at least one of the x_k', y_k' is irrational (why?)

The corresponding Riemann sum equals

$$S_n'(f,P) = \sum_{k=1}^{n} f(x_k', y_k') \Delta A_k = \sum_{k=1}^{n} 1 \cdot \Delta A_k = 1$$

$$\to 1, \quad \text{as } \|P\| \to 0$$

Since $S_n(f,P) \to 0 \neq 1 \leftarrow S_n'(f,P)$,

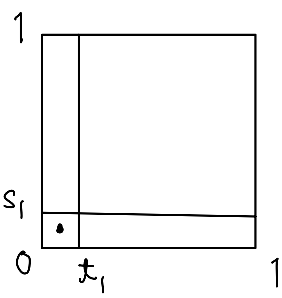
f is not integrable. ※

<u>eg6</u>: Let $R = [0,1] \times [0,1]$

$$f(x,y) = \begin{cases} \dfrac{1}{xy}, & \text{if } x \neq 0 \text{ and } y \neq 0 \\ 0, & \text{if } x = 0 \text{ or } y = 0 \end{cases} \qquad \begin{pmatrix} \geq 0 \\ \text{on } R \end{pmatrix}$$

Then f is <u>not</u> integrable over R. (using (i))

<u>Soln</u> In any partition P of R,



there is a sub-rectangle

$R_1 = [0, t_1] \times [0, s_1]$

Choose $(x_1, y_1) = (t_1^2, s_1^2) \in R_1$

(since $0 < t_1^2 < t_1 < 1$, $0 < s_1^2 < s_1 < 1$)

Then Riemann sum

$$S(f,P) = \sum_{k=1}^{n} f(x_k, y_k) \Delta A_k$$

$$= f(x_1,y_1)\Delta A_1 + \underbrace{\sum_{k=2}^{n} f(x_k,y_k)\Delta A_k}_{\geq 0}$$

$$\geq f(t_1^2, s_1^2)\, t_1 s_1$$

$$= \frac{1}{t_1^2 s_1^2}\, t_1 s_1 = \frac{1}{t_1 s_1}$$

Since $0 < t_1, s_1 \leq \|P\| \to 0$, $t_1, s_1 \to 0$

Hence $S(f,P) \geq \dfrac{1}{t_1 s_1} \to \infty$ as $\|P\| \to 0$.

$\therefore$ Limit doesn't exist, & f is not integrable. ※

<u>Remark:</u> Egs 5 & 6 show that we need "<u>condition(s)</u>" to ensure the <u>integrability</u> of a function over closed rectangle.

<u>Prop1</u>: Let $R = [a,b] \times [c,d]$ be a <u>closed rectangle</u>, and $f(x,y)$ be an <u>integrable</u> function over R, then f is <u>bounded</u> on R.

(i.e. $\exists\, M > 0$ such that "$|f(x,y)| \leq M$, $\forall (x,y) \in R$.")

<u>Pf</u>: Omitted (eg6 above gives an idea of proof.)

<u>Remark</u>: From eg 5, "boundedness" is necessary, but <u>not</u> sufficient for integrability.

$$\boxed{\begin{array}{c} \text{integrable} \Rightarrow \text{bounded} \\ \not\Leftarrow \\ (\Leftarrow \text{ in general}) \end{array}}$$

<u>Prop 2</u>: Let $R = [a,b] \times [c,d]$ be a <u>closed rectangle</u>, and $f(x,y)$ be a <u>continuous</u> function on R, then f is <u>integrable</u> on R.

<u>Pf</u>: Omitted (See proof in 1-variable case in MATH 2060 for an idea of proof.)

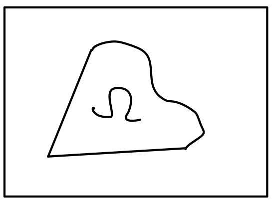
<u>Remarks</u>: (i) Note that a <u>continuous</u> function on <u>closed rectangle</u> is always <u>bounded</u> (Props 1 & 2 are consistent) (MATH 2050 for 1-variable situation)

(ii) "continuity" (on closed rectangle) is <u>sufficient</u>, but <u>not necessary</u>. In fact, Prop 2 can be generalized to a <u>bounded</u> function on a closed rectangle with a "small" set of <u>discontinuity</u>. The precise concept is "measure zero set" (MATH 4050 Real Analysis).

For us, we have

> **Prop 2'** For function over <u>closed rectangle</u>
>
> (a) bounded + "continuous except finitely many points"
>
> $\Longrightarrow$ integrable.
>
> (b) bounded + "continuous except finitely many differentiable curves"
>
> $\Rightarrow$ integrable

egs (i)



$$f(x,y) = \begin{cases} \text{continuous on } \Omega \text{ (and bounded)} \\ 0, \quad \text{otherwise (outside } \Omega) \end{cases}$$

(ii)