Homework 6

Solutions

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(6.1) In this question, consider the following function $f : \mathbb{R} \to [0, \infty)$

$$f(t) = \begin{cases} \exp(-\frac{1}{t}) & : t > 0 \\ 0 & : t \le 0 \end{cases}.$$

a) Show that $f(t)$ is a smooth function.

b) Calculate the k^{th}-order Taylor polynomial of $f(t)$ at $t = 0$ for any $k \in \mathbb{N}$.

c) Define the function

$$F(x) = \frac{f(2 - \|x\|)}{f(2 - \|x\|) + f(\|x\| - 1)}, \quad \forall x \in \mathbb{R}^n.$$

Show that F is a smooth function on $\mathbb{R}^n$ with

$$0 \le F(x) \le 1, \quad \forall x \in \mathbb{R}^n.$$

Moreover, show that $F(x) = 1$ if $\|x\| \le 1$, and $F(x) = 0$ if $\|x\| \ge 2$.

Solution (6.1)

a) By repeatedly applying the Chain Rule, we see that the composition of smooth functions is smooth. Since $-\frac{1}{t}$ is smooth for $t > 0$ and e^t is smooth on $\mathbb{R}$, by the Chain rule, $f(t)$ is smooth for $t > 0$. Similarly, $f(t) \equiv 0$ is smooth for $t < 0$. Therefore, to show $f(t)$ is smooth on $\mathbb{R}$ it suffices to show that for each $n \in \mathbb{N}_0$,

$$\lim_{t \downarrow 0} \exp(-t^{-1}) \cdot t^{-n} = 0.$$

For $n = 0$, the limit is clearly zero. By L'Hôpital's rule

$$\lim_{t \downarrow 0} \frac{t^{-(n+1)}}{\exp(t^{-1})} = \lim_{t \downarrow 0} \frac{-(n+1)t^{-(n+2)}}{-t^{-2}\exp(t^{-1})} = (n+1) \lim_{t \downarrow 0} \frac{t^{-n}}{\exp(t^{-1})},$$

and we are done by induction.

b) Since the function and all its derivatives vanish at zero, $P_k(t) \equiv 0$ for any $k \in \mathbb{N}$.

c) Since $f \ge 0$, in order for the denominator to be zero we require that $f(2-\|x\|) = f(\|x\| - 1) = 0$. But this is only true if both $2 - \|x\| \le 0$ and $\|x\| - 1 \le 0$, which is impossible. Therefore, by the composition of smooth functions being smooth, the function $F(x)$ is well-defined and smooth everywhere on $\mathbb{R}^n$. Also, we note that $f \ge 0$ implies $0 \le F \le 1$.

Finally, for $\|x\| \leq 1$ or $\|x\| - 1 \leq 0$, we have

$$F(x) = \frac{f(2 - \|x\|)}{f(2 - \|x\|) + f(\|x\| - 1)} = \frac{f(2 - \|x\|)}{f(2 - \|x\|)} = 1.$$

For $\|x\| \geq 2$ or $2 - \|x\| \leq 0$, we have

$$F(x) = \frac{f(2 - \|x\|)}{f(2 - \|x\|) + f(\|x\| - 1)} = \frac{0}{f(\|x\| - 1)} = 0.$$

(6.2)

a) Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a continuous function such that $\lim_{x\to\infty} f(x) = \infty$. That is

$$\forall C \in \mathbb{R},\ \exists R > 0 \quad \text{such that} \quad \|x\| \geq R \implies f(x) > C.$$

Show that f attains a global minimum on $\mathbb{R}^n$.

b) Suppose $g : \mathbb{R}^n \to (0, \infty)$ is a positive continuous function such that $\lim_{x\to\infty} g(x) = 0$. That is

$$\forall \epsilon > 0,\ \exists R > 0 \quad \text{such that} \quad \|x\| \geq R \implies g(x) < \epsilon.$$

Show that g attains a global maximum on $\mathbb{R}^n$.

c) Does the function $g : \mathbb{R}^n \to (0, \infty)$ from part b) necessarily attain a global minimum? Justify your answer.

d) Find the global maximum of the function

$$h(x, y) = \frac{1 + |x| + |y|}{1 + x^2 + y^2}, \quad \forall (x, y) \in \mathbb{R}^2.$$

Solution (6.2)

a) By our assumption on f, we may find $R > 0$ such that $f(x) \geq f(0)$ when $\|x\| \geq R$. Let $A = \overline{B_R(0)}$. By EVT, $f|_A$ has a global minimum at $x_0 \in A$. Since $f(x) \geq f(0) \geq f(x_0)$ for any x outside of A, f has a global minimum at x_0.

b) By our assumption on g, we find $R > 0$ such that $g(x) < g(0)$ when $\|x\| \geq R$. Again, let $A = \overline{B_R(0)}$. By EVT, $g|_A$ has a global maximum at $x_0 \in A$. Since $g(x) < g(0) \leq g(x_0)$ for any x outside of A, f has a global maximum at x_0.

c) No. Consider the function $g : \mathbb{R} \to \mathbb{R}$, $g(t) = (1 + t^2)^{-1}$.

d) $h : \mathbb{R}^2 \to \mathbb{R}$ is a continuous positive function such that $\lim_{(x,y)\to\infty} h(x, y) = 0$, so by part b), h attains a global maximum. The function h is not differentiable at the coordinate axes. Away from the coordinate axes, by symmetry, we may assume $x, y > 0$, and that

$$\nabla h(x,y) = \left(\frac{(1+x^2+y^2) - 2x(1+x+y)}{(1+x^2+y^2)^2}, \frac{(1+x^2+y^2) - 2y(1+x+y)}{(1+x^2+y^2)^2} \right)$$
$$= \left(\frac{1-x^2+y^2-2xy-2x}{(1+x^2+y^2)^2}, \frac{1+x^2-y^2-2xy-2y}{(1+x^2+y^2)^2} \right)$$

and hence $\nabla h = 0$ iff

$$1 - x^2 + y^2 - 2xy - 2x = 0, \quad 1 + x^2 - y^2 - 2xy - 2y = 0.$$

Subtracting these two equations from one another gives

$$x^2 - y^2 + x - y = (x-y)(x+y+1) = 0,$$

which in the region $x, y > 0$ has the unique solution $x = y$. Substituting this back into our original equations, we find $2x^2 + 2x - 1 = 0$, which we solve to find $x = \frac{\sqrt{3}-1}{2}$ (as we assumed $x > 0$). Therefore, h has the four critical points

$$(x,y) = (\pm\frac{\sqrt{3}-1}{2}, \pm\frac{\sqrt{3}-1}{2}),$$

away from the coordinate axes.

To analyse the critical points on the coordinate axes, from the symmetry $h(x,0) = h(0,x)$, it suffices to consider the maximum of h restricted to the x-axis:

$$\tilde{h}(x) = h(x,0) = \frac{1+|x|}{1+x^2}, \quad \forall x \in \mathbb{R}.$$

$\tilde{h}$ is not differentiable at 0 with $\tilde{h}(0) = 1$, and away from zero we have

$$\frac{d\tilde{h}}{dx}(x) = \frac{\text{sign}(x)(1-x^2) - 2x}{(1+x^2)^2}.$$

Since $\tilde{h}$ is an even function, we only need to consider the case $x > 0$, and so $\frac{d\tilde{h}}{dx}(x) = 0$ iff $x^2 + 2x - 1 = (x+1-\sqrt{2})(x+1+\sqrt{2}) = 0$. Since $x > 0$, the only solution is $x = \sqrt{2} - 1$. Note that

$$\tilde{h}(\sqrt{2}-1) = \frac{1}{2\sqrt{2}-2} > 1 = \tilde{h}(0),$$

so the maximum of h on the coordinate axes is $\frac{1}{2\sqrt{2}-2} \approx 1.207$. However, at our original critical points, we find

$$h\left(\pm\frac{\sqrt{3}-1}{2}, \pm\frac{\sqrt{3}-1}{2}\right) = \frac{1}{\sqrt{3}-1} \approx 1.366,$$

and therefore, h has maxima at the points $(x_0, y_0) = \left(\pm\frac{\sqrt{3}-1}{2}, \pm\frac{\sqrt{3}-1}{2}\right)$, with maximal value $\frac{1}{\sqrt{3}-1}$ at these points.

(6.3) Consider the function $F : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$F(x, y) = \sin(x)\sin(y), \quad \forall (x, y) \in \mathbb{R}^2.$$

a) Find and classify the critical points of F.

b) At each critical point, find the 2^{nd}-order Taylor polynomial P_2.

Solution (6.3)

a) We note that F is a C^1 function on $\mathbb{R}^2$ with

$$\nabla F(x, y) = (\cos x \sin y, \sin x \cos y).$$

Therefore $\nabla F(x, y) = 0$ iff $\sin x = \sin y = 0$ or $\cos x = \cos y = 0$, which happens iff $x, y \in \pi\mathbb{Z}$ or $x, y \in \frac{\pi}{2} + \pi\mathbb{Z}$. The Hessian matrix is given by

$$HF(x, y) = \begin{pmatrix} -\sin x \sin y & \cos x \cos y \\ \cos x \cos y & -\sin x \sin y \end{pmatrix}.$$

At the first type of critical point $x, y \in \pi\mathbb{Z}$, we find that

$$HF(x, y) = \begin{pmatrix} 0 & \cos x \cos y \\ \cos x \cos y & 0 \end{pmatrix}.$$

This has negative determinant, and is hence a saddle point. At the second type of critical point $x, y \in \frac{\pi}{2} + \pi\mathbb{Z}$, we find that

$$HF(x, y) = \begin{pmatrix} -\sin x \sin y & 0 \\ 0 & -\sin x \sin y \end{pmatrix}.$$

This has positive determinant $\sin^2 x \sin^2 y > 0$. We divide these critical points further. Note that $\sin x, \sin y \in \{\pm 1\}$. If $\sin x = \sin y$, then their product is 1 and the critical point is a local maximum. If $\sin x = -\sin y$, then their product is -1 and the critical point is a local minimum.

b) Consider the saddle points $(\alpha\pi, \beta\pi)$ with $\alpha, \beta \in \mathbb{Z}$ and $\alpha + \beta$ even. Then $\cos(\alpha\pi)\cos(\beta\pi) = 1$, and

$$F(\alpha\pi, \beta\pi) = 0, \quad HF(\alpha\pi, \beta\pi) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Hence the 2^{nd}-Taylor polynomial of F at $(\alpha\pi, \beta\pi)$ is

$$P_2(x, y) = (x - \alpha\pi)(y - \beta\pi).$$

At the other saddle points $(\alpha\pi, \beta\pi)$ with $\alpha, \beta \in \mathbb{Z}$ and $\alpha + \beta$ odd, we have $\cos(\alpha\pi)\cos(\beta\pi) = -1$, and

$$F(\alpha\pi, \beta\pi) = 0, \quad HF(\alpha\pi, \beta\pi) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

Hence the 2^{nd}-Taylor polynomial of F at $(\alpha\pi, \beta\pi)$ is

$$P_2(x,y) = -(x-\alpha\pi)(y-\beta\pi).$$

Next, consider the local maxima $(\alpha\pi + \frac{\pi}{2}, \beta\pi + \frac{\pi}{2})$ with $\alpha, \beta \in \mathbb{Z}$ and $\alpha + \beta$ even. Then $\sin(\alpha\pi + \frac{\pi}{2})\sin(\beta\pi + \frac{\pi}{2}) = 1$, and

$$F(\alpha\pi + \frac{\pi}{2}, \beta\pi + \frac{\pi}{2}) = 1, \quad HF(\alpha\pi + \frac{\pi}{2}, \beta\pi + \frac{\pi}{2}) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Hence the 2^{nd}-Taylor polynomial of F at $(\alpha\pi + \frac{\pi}{2}, \beta\pi + \frac{\pi}{2})$ is

$$P_2(x,y) = 1 - \frac{1}{2}(x - (\alpha\pi + \frac{\pi}{2}))^2 - \frac{1}{2}(y - (\beta\pi + \frac{\pi}{2}))^2.$$

Finally consider the local minima $(\alpha\pi + \frac{\pi}{2}, \beta\pi + \frac{\pi}{2})$ with $\alpha, \beta \in \mathbb{Z}$ and $\alpha + \beta$ odd. Then $\sin(\alpha\pi + \frac{\pi}{2})\sin(\beta\pi + \frac{\pi}{2}) = -1$, and

$$F(\alpha\pi + \frac{\pi}{2}, \beta\pi + \frac{\pi}{2}) = -1, \quad HF(\alpha\pi + \frac{\pi}{2}, \beta\pi + \frac{\pi}{2}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence the 2^{nd}-Taylor polynomial of F at $(\alpha\pi + \frac{\pi}{2}, \beta\pi + \frac{\pi}{2})$ is

$$P_2(x,y) = -1 + \frac{1}{2}(x - (\alpha\pi + \frac{\pi}{2}))^2 + \frac{1}{2}(y - (\beta\pi + \frac{\pi}{2}))^2.$$