Homework 5

Solutions

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(5.1) In each of the following examples, find the partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ at the point (x_0, y_0, z_0):

a)
$$f(t) = e^t, \quad (x_0, y_0, z_0) = (0, 0, 0), \quad t = (x + z)e^y.$$

b)
$$f(u, v) = \arctan(uv), \quad (x_0, y_0, z_0) = (0, 0, 0),$$
$$u = x \sin y \sin z + x \cos z, \quad v = x \cos y \sin z.$$

c)
$$f(a, b, c, d) = a^2 b + c^2 d - e^{ac} \log(bd), \quad (x_0, y_0, z_0) = (1, 1, 1),$$
$$a = x + y + z, \quad b = x^2, \quad c = y^2, \quad d = z^2.$$

Solution (5.1)

a) We have the partial derivatives

$$f_t = e^t, \quad t_x = e^y, \quad t_y = (x + z)e^y, \quad t_z = e^y.$$

At $(x_0, y_0, z_0) = (0, 0, 0)$, $t = 0$ and we have

$$f_t = 1, \quad t_x = 1, \quad t_y = 0, \quad t_z = 1.$$

Applying the Chain Rule, we conclude $(f_x, f_y, f_z) = (1, 0, 1)$.

b) Note that u_y, u_z, v_y, v_z all contain a factor of x, and thus at $x_0 = 0$, these partial derivative are all zero. Therefore, $f_y = f_z = 0$ at $(x_0, y_0, z_0) = (0, 0, 0)$. By the Chain Rule

$$\begin{aligned} f_x &= f_u \cdot u_x + f_v \cdot v_x \\ &= f_u(\sin y \sin z + \cos z) + f_v(\cos y \sin z) \end{aligned}$$

At the point $(x_0, y_0, z_0) = (0, 0, 0)$, $(u, v) = (0, 0)$ and hence

$$f_x|_{(0,0,0)} = f_u(0, 0) = \frac{v}{1 + u^2 v^2}|_{(0,0)} = 0.$$

c) By a direct calculation

$$Df(a, b, c, d) = (2ab - ce^{ac} \log(bd), a^2 - e^{ac} b^{-1}, 2cd - ae^{ac} \log(bd), c^2 - e^{ac} d^{-1}),$$

and setting $g = (a, b, c, d)$, we have

$$Dg(x, y, z) = \begin{pmatrix} 1 & 1 & 1 \\ 2x & 0 & 0 \\ 0 & 2y & 0 \\ 0 & 0 & 2z \end{pmatrix}.$$

At $(x_0, y_0, z_0) = (1, 1, 1)$, $(a, b, c, d) = (3, 1, 1, 1)$ and we find that

$$Df(3, 1, 1, 1) = (6, 9 - e^3, 2, 1 - e^3),$$

$$Dg(1, 1, 1) = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

and hence

$$\nabla f(1, 1, 1) = Df(3, 1, 1, 1) \cdot Dg(1, 1, 1) = (24 - 2e^3, 10, 8 - 2e^3).$$

(5.2) Consider the curve determined by the equation

$$xe^y + \sin(xy) + y = \log 2.$$

a) At which points does the curve intersect the horizontal line $y = \log 2$.

b) Assume that y is locally given as a function of x around each of the points found in part a). Find the value of $\frac{dy}{dx}$ at these points.

Solution (5.2)

a) Subbing $y = \log 2$ into our equation, we find that

$$2x + \sin(x \log 2) = 0.$$

We find the obvious solution $x = 0$. To see that there are not any others, we notice that for $|x| > \frac{1}{2}$,

$$|2x + \sin(x \log 2)| \geq 2\,|x| - |\sin(x \log 2)| > 1 - 1 = 0,$$

and so there can only be solutions in the range $x \in [-\frac{1}{2}, \frac{1}{2}]$. However, for any x in this range, the sign on $\sin(x \log 2)$ is the same as the sign of x, and hence there are no non-zero solutions.

b) By applying the Chain Rule, we find

$$e^y + xe^y \frac{dy}{dx} + \cos(xy)(y + x\frac{dy}{dx}) + \frac{dy}{dx} = 0.$$

At the point $(x, y) = (0, \log 2)$, this equation becomes

$$2 + 0 + \log 2 + \frac{dy}{dx} = 0,$$

and hence $\frac{dy}{dx}|_{(0,\log 2)} = -2 - \log 2$.

(5.3) Suppose $f : \Omega_1 \subseteq \mathbb{R}^n \to \Omega_2 \subseteq \mathbb{R}^n$ is a bijection, with inverse $f^{-1} : \Omega_2 \to \Omega_1$.

a) Let $I : \mathbb{R}^n \to \mathbb{R}^n$ be the identity map. Calculate its Jacobian matrix at any point of $\mathbb{R}^n$.

b) Suppose f is differentiable at $a \in \Omega_1$ and f^{-1} is differentiable at $f(a) \in \Omega_2$. Express the Jacobian matrix of f^{-1} at $f(a)$ in terms of $Df(a)$.

c) Consider the particular example of the smooth bijection $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$. Using part b), calculate the derivative of $f^{-1}(y)$ at $y = 1$.

d) Explain why we cannot use part b) when $x = 0$.

Solution (5.3)

a) $I_i(x_1, \ldots, x_n) = x_i$, and so $\frac{\partial I_i}{\partial x_j} = \delta_{ij}$. That is, $DI(x) = I$ for any $x \in \mathbb{R}^n$.

b) Applying the Chain Rule to $I = f^{-1} \circ f$ at a yields

$$I = DI(x) = Df^{-1}(f(a)) \cdot Df(a).$$

In particular, $Df(a)$ is invertible, and so $Df^{-1}(f(a)) = (Df(a))^{-1}$.

c) Since $f(1) = 1$ and $Df(1) = 3$, we find that

$$Df^{-1}(1) = Df^{-1}(f(1)) = Df(1)^{-1} = \frac{1}{3}.$$

d) f^{-1} is not differentiable at 0, and so the Chain Rule doesn't hold.

(5.4) Consider a curve $\gamma : I \to \mathbb{R}^3$ parameterised in Spherical coordinates

$$\gamma(t) = (\rho(t), \varphi(t), \theta(t)), \quad t \in I.$$

a) Using the formulas for the Cartesian coordinates

$$\begin{aligned} x &= \rho \sin\varphi \cos\theta, \\ y &= \rho \sin\varphi \sin\theta, \\ z &= \rho \cos\varphi, \end{aligned}$$

find equations for $x'(t)$, $y'(t)$, and $z'(t)$ in terms of $\rho'(t)$, $\varphi'(t)$, and $\theta'(t)$.

b) Substituting your answers from a) into the formula

$$\|\gamma'(t)\| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2},$$

find an equation for the speed of the curve in spherical coordinates.

c) Using the formula from b) or otherwise, calculate the arclength of the curve

$$\begin{cases} \rho(t) = & e^t, \\ \theta(t) = & \log(\tan t), \quad \forall t \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right). \\ \varphi(t) = & 2t, \end{cases}$$

Solution (5.4)

a) By the Chain Rule, we have

$$\begin{aligned}
\dot{x} &= \dot{\rho}\sin\varphi\cos\theta + \rho\cos\varphi\dot{\varphi}\cos\theta - \rho\sin\varphi\sin\theta\dot{\theta} \\
\dot{y} &= \dot{\rho}\sin\varphi\sin\theta + \rho\cos\varphi\dot{\varphi}\sin\theta + \rho\sin\varphi\cos\theta\dot{\theta} \\
\dot{z} &= \dot{\rho}\cos\varphi - \rho\sin\varphi\dot{\varphi}
\end{aligned}$$

b) We note that

$$\begin{aligned}
\dot{x}^2 &= \dot{\rho}^2\sin^2\varphi\cos^2\theta + \rho^2\cos^2\varphi\dot{\varphi}^2\cos^2\theta + \rho^2\sin^2\varphi\sin^2\theta\dot{\theta}^2 \\
&\quad + 2\rho\dot{\rho}\sin\varphi\cos\varphi\dot{\varphi}\cos^2\theta - 2\rho\dot{\rho}\sin^2\varphi\cos\theta\sin\theta\dot{\theta} \\
&\quad - 2\rho^2\cos\varphi\sin\varphi\dot{\varphi}\cos\theta\sin\theta\dot{\theta}, \\
\dot{y}^2 &= \dot{\rho}^2\sin^2\varphi\sin^2\theta + \rho^2\cos^2\varphi\dot{\varphi}^2\sin^2\theta + \rho^2\sin^2\varphi\cos^2\theta\dot{\theta}^2 \\
&\quad + 2\rho\dot{\rho}\sin\varphi\cos\varphi\dot{\varphi}\sin^2\theta + 2\rho\dot{\rho}\sin^2\varphi\sin\theta\cos\theta\dot{\theta} \\
&\quad + 2\rho^2\cos\varphi\sin\varphi\dot{\varphi}\sin\theta\cos\theta\dot{\theta}, \\
\dot{z}^2 &= \dot{\rho}^2\cos^2\varphi + \rho^2\sin^2\varphi\dot{\varphi}^2 - 2\rho\dot{\rho}\cos\varphi\sin\varphi\dot{\varphi}.
\end{aligned}$$

Summing these together we have

$$\|\gamma'(t)\| = \sqrt{\dot{\rho}^2 + \rho^2\dot{\varphi}^2 + \rho^2\sin^2(\varphi)\dot{\theta}^2}.$$

c) From part b) the arc length is given by the formula

$$S = \int_{\pi/4}^{\pi/2} \sqrt{\dot{\rho}^2 + \rho^2\dot{\varphi}^2 + \rho^2\sin^2(\varphi)\dot{\theta}^2}\,dt$$

Note that $\dot{\rho} = e^t$, $\dot{\varphi} = 2$, $\dot{\theta} = 2\csc(2t)$. Therefore,

$$\begin{aligned}
S &= \int_{\pi/4}^{\pi/2} \sqrt{e^{2t} + 4e^{2t} + 4e^{2t}\sin^2(2t)\csc^2(2t)}\,dt \\
&= \int_{\pi/4}^{\pi/2} 3e^t\,dt \\
&= 3(e^{\pi/2} - e^{\pi/4}).
\end{aligned}$$