Homework 4

Solutions

. . . .

(4.1) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \frac{x^3}{x^2+y^2} & : (x,y) \neq (0,0) \\ 0 & : (x,y) = (0,0) \end{cases}.$$

a) Calculate the directional derivative $D_u f(0,0)$ for the vector $u = (\cos\theta, \sin\theta)$.

b) Calculate the gradient $\nabla f(0,0)$.

c) Is the function f differentiable at the origin? Justify your answer.

Solution (4.1)

a) Since

$$f(hu) - f(0) = \frac{h^3\cos^3\theta}{h^2(\cos^2\theta + \sin^2\theta)} - 0 = h\cos^3\theta,$$

we have

$$D_u f(0,0) = \lim_{h\to 0} \frac{f(hu) - f(0)}{h} = \lim_{h\to 0} \cos^3\theta = \cos^3\theta.$$

b)

$$\begin{aligned} \nabla f(0,0) &= \big(D_{(1,0)} f(0,0), D_{(0,1)} f(0,0)\big) \\ &= \left(\cos^3(0), \cos^3\left(\frac{\pi}{2}\right)\right) = (1,0). \end{aligned}$$

c) If f was differentiable at the origin, then

$$\cos^3\theta = D_u f(0,0) = \nabla f(0,0)\cdot u = (1,0)\cdot(\cos\theta, \sin\theta) = \cos\theta,$$

but this is clearly not true. Therefore, f is **not** differentiable at the origin.

(4.2) Consider the family of functions

$$f_\alpha(x,y,z) = e^x + y\cos z + \alpha(x^3 - y + e^{-z}\sin x), \quad \forall \alpha \in \mathbb{R}.$$

For which values of $\alpha \in \mathbb{R}$ does the function increase most rapidly in a direction parallel to the x-axis at the origin? Justify your answer.

Solution (4.2) We need to find those values of $\alpha \in \mathbb{R}$ such that $\nabla f_\alpha(0,0,0)$ is parallel to the x-axis. We note that

$$\begin{aligned}\frac{\partial f_\alpha}{\partial x} &= e^x + \alpha(3x^2 + e^{-z}\cos x),\\ \frac{\partial f_\alpha}{\partial y} &= \cos z - \alpha,\\ \frac{\partial f_\alpha}{\partial z} &= -y\sin z - \alpha e^{-z}\sin x,\end{aligned}$$

and so

$$\frac{\partial f_\alpha}{\partial x}(0,0,0) = 1+\alpha, \quad \frac{\partial f_\alpha}{\partial y}(0,0,0) = 1-\alpha, \quad \frac{\partial f_\alpha}{\partial z}(0,0,0) = 0.$$

Therefore $\nabla f_\alpha(0,0,0) = (1+\alpha, 1-\alpha, 0)$ is parallel to the x-axis if and only if $\alpha = 1$.

(4.3) Let $\Omega \subseteq \mathbb{R}^n$ open, $f : \Omega \to \mathbb{R}$ and $a \in \Omega$. We assume that there exists some affine function $L : \mathbb{R}^n \to \mathbb{R}$, defined by

$$L(x) = \lambda + \alpha \cdot (x - a), \quad (\lambda \in \mathbb{R},\ \alpha \in \mathbb{R}^n)$$

such that the error function $\varepsilon(x) = f(x) - L(x)$ satisfies

$$\lim_{x\to a} \frac{\varepsilon(x)}{\|x-a\|} = 0.$$

a) Show that $\lim\limits_{x\to a} f(x)$ exists and equals λ.

b) Suppose also that all the partial derivatives $\frac{\partial f}{\partial x_i}(a)$ (for $1 \le i \le n$) exist.

Show that f is continuous at a.

c) With our assumption from part b), conclude that f is differentiable at a.

Solution (4.3)

a) Since $\lim_{x\to a} \|x-a\| = 0$, by the product of limits

$$\lim_{x\to a} \varepsilon(x) = \lim_{x\to a} \frac{\varepsilon(x)}{\|x-a\|} \cdot \lim_{x\to a} \|x-a\| = 0.$$

Therefore

$$\begin{aligned}\lim_{x\to a} f(x) &= \lim_{x\to a} L(x) + \varepsilon(x)\\ &= \lim_{x\to a} L(x) + \lim_{x\to a} \varepsilon(x) = \lambda + 0.\end{aligned}$$

b) Fix $1 \leq i \leq n$. We first note that

$$\begin{aligned}
\lim_{h \to 0} \frac{f(a + he_i) - \lambda}{h} &= \lim_{h \to 0} \frac{L(a + he_i) - \lambda + \epsilon(a + he_i)}{h} \\
&= \lim_{h \to 0} \frac{h(\alpha \cdot e_i) + \epsilon(a + he_i)}{h} \\
&= \alpha \cdot e_i + \underbrace{\lim_{h \to 0} \frac{\epsilon(a + he_i)}{h}}_{=0} \\
&= \alpha \cdot e_i \in \mathbb{R},
\end{aligned}$$

is a well-defined limit. By our assumption

$$\lim_{h \to 0} \frac{f(a + he_i) - f(a)}{h} = \frac{\partial f}{\partial x_i}(a) \in \mathbb{R},$$

is also a well-defined limit, and therefore

$$\begin{aligned}
\lim_{h \to 0} \frac{f(a) - \lambda}{h} &= \lim_{h \to 0} \frac{(f(a + he_i) - \lambda) - (f(a + he_i) - f(a))}{h} \\
&= \lim_{h \to 0} \frac{f(a + he_i) - \lambda}{h} - \lim_{h \to 0} \frac{f(a + he_i) - f(a)}{h} \\
&= \alpha \cdot e_i - \frac{\partial f}{\partial x_i}(a) \in \mathbb{R}.
\end{aligned}$$

Since the numerator is a constant, this is only possible if the numerator is actually equal to zero, and hence $\lambda = f(a)$ and f is continuous at a.

c) From the previous argument, we also deduce that $\alpha \cdot e_i = \frac{\partial f}{\partial x_i}(a)$ for each $1 \leq i \leq n$. In particular $\alpha = \nabla f(a)$ and

$$L(x) = f(a) + \nabla f(a) \cdot (x - a),$$

which is precisely the definition of f being differentiable at a.

Remark: This question shows that if all the partial derivatives of f exist at a point, then the *only* possible affine approximation of f to first order is the one given in the definition of differentiability.

(4.4) An important partial differential equation that describes the distribution of heat in a region at time t can be represented by the one-dimensional heat equation

$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2}.$$

For constants $\alpha, \beta \in \mathbb{R}$, consider the function

$$u(x, t) := \sin(\alpha x) e^{-\beta t}.$$

Find a relationship between the constants α and β for this function to be a solution to the one-dimensional heat equation.

Solution (4.4) We note that

$$\begin{aligned}\frac{\partial u}{\partial t} &= -\beta \sin(\alpha x)e^{-\beta t} = -\beta u(x,t),\\ \frac{\partial u}{\partial x} &= \alpha \cos(\alpha x)e^{\beta t},\\ \frac{\partial^2 u}{\partial x^2} &= -\alpha^2 \sin(\alpha x)e^{\beta t} = -\alpha^2 u(x,t),\end{aligned}$$

and therefore u solves the heat equation iff $\beta = \alpha^2$.