Homework 3

Solutions

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(3.1)

a) Let $x_0 \in \mathbb{R}^n$ and $r > 0$. Show that the set

$$B_r(x_0) = \{x \in \mathbb{R}^n : \|x - x_0\| < r\},$$

is an open set.

This justifies naming $B_r(x_0)$ an open ball.

b) Show that, for a pair of open sets X_1, X_2, their union $X_1 \cup X_2$ is also an open set.

c) Show that, for a pair of open sets X_1, X_2, their intersection $X_1 \cap X_2$ is also an open set.

d) Suppose now we have a (not necessarily finite) collection of open sets

$$\{X_\alpha : \alpha \in \mathcal{A}\}.$$

Is their union $U := \bigcup_{\alpha \in \mathcal{A}} X_\alpha$ necessarily open? Justify your answer.

e) Is their intersection $C := \bigcap_{\alpha \in \mathcal{A}} X_\alpha$ necessarily open? Justify your answer.

Solution (3.1)

a) Fix $y \in B_r(x_0)$. Let $\epsilon = r - \|x_0 - y\| > 0$. Then, for any point $z \in B_\epsilon(y)$, we have

$$\|z - x_0\| \leq \|z - y\| + \|y - x_0\| < \epsilon + \|y - x_0\| = r,$$

and hence $B_\epsilon(y) \subseteq B_r(x_0)$. So $B_r(x_0)$ is open.

b) If $x \in X_1 \cup X_2$, then either $x \in X_1$ or $x \in X_2$. If $x \in X_1$, then since X_1 is open, we can find $\epsilon > 0$ such that

$$B_\epsilon(x) \subseteq X_1 \subseteq X_1 \cup X_2.$$

The same holds if $x \in X_2$.

c) If $x \in X_1 \cap X_2$, then $x \in X_1$ and $x \in X_2$. Since X_1 and X_2 are open, there exists $\epsilon_1, \epsilon_2 > 0$ such that

$$B_{\epsilon_1}(x) \subseteq X_1, \quad B_{\epsilon_2}(x) \subseteq X_2.$$

Therefore, setting $\epsilon = \min\{\epsilon_1, \epsilon_2\} > 0$, we can

$$B_\epsilon(x) \subseteq X_1, \quad B_\epsilon(x) \subseteq X_2,$$

and therefore $B_\epsilon(x) \subseteq X_1 \cap X_2$.

d) If $x \in U$, there exists $\alpha \in \mathcal{A}$ such that $x \in X_\alpha$, and therefore, for some $\epsilon > 0$,

$$x \in B_\epsilon(x) \subseteq X_\alpha \subseteq U,$$

so U is open.

e) C is not necessarily open. Consider $X_k := B_{\frac{1}{k}}(0) \subseteq \mathbb{R}^n$ for $k \in \mathbb{N}$. Then each X_k is open, but $C = \{0\}$ is not open.

(3.2) Let $A \subseteq \mathbb{R}^n$ and consider the set of vectors whose distance to A is zero:

$$\{x \in \mathbb{R}^n : \forall \epsilon > 0, \exists a \in A \text{ such that } \|x - a\| < \epsilon\}.$$

Show that this set is the same as the closure of A:

$$\overline{A} = \text{Int}(A) \cup \partial A.$$

Solution (3.2) If $x \in \text{Ext}(A)$, then there exists some $\epsilon > 0$ such that $B_\epsilon(x) \subseteq \mathbb{R}^n \setminus A$, and hence $\|x - a\| \geq \epsilon$ for all $a \in A$. So the set of vectors of distance zero to A is a subset of $\overline{A}$. To show every point of $x \in \overline{A}$ has distance zero to A, we note that for any $\epsilon > 0$, $B_\epsilon(x) \cap A \neq \emptyset$, and so we can find some $a \in A \cap B_\epsilon(x)$. In particular, we have found $a \in A$ with $\|x - a\| < \epsilon$.

(3.3) For the following examples, does the limit exist, and if so, what is its value?

a) $\lim\limits_{(x,y)\to(0,0)} \frac{\sin(xy)}{xy}$

b) $\lim\limits_{(x,y)\to(0,0)} \frac{\sin(xy)}{x+y}$

c) $\lim\limits_{(x,y)\to(0,0)} \frac{\sin(xy)}{x^2+y^2}$

Solution (3.3)

a) By L'Hôpital's rule, $\lim_{z\to 0} \frac{\sin z}{z} = \lim_{z\to 0} \frac{\cos z}{1} = 1$. Therefore, if we fix $\epsilon > 0$, then there exists some $\eta > 0$ such that if $0 < |xy| < \eta$, then $\left|\frac{\sin(xy)}{xy} - 1\right| < \epsilon$. Furthermore, we have the inequality

$$|xy| \leq \frac{1}{2}(x^2 + y^2) = \frac{1}{2}\|(x,y)\|^2.$$

In particular, choosing $\delta = \sqrt{\eta} > 0$, we find that if if $0 < \|(x,y)\| < \delta$ and both $x, y \neq 0$, then $0 < |xy| < \eta$, and hence $\left|\frac{\sin(xy)}{xy} - 1\right| < \epsilon$. That is, $\lim\limits_{(x,y)\to(0,0)} \frac{\sin(xy)}{xy} = 1$.

b) Along the x-axis, we have

$$\lim_{(x,0)\to(0,0)} \frac{\sin(xy)}{x+y} = \lim_{x\to 0} \frac{\sin(0)}{x} = 0.$$

However, along the curve $y = x^2 - x$ we apply L'Hôpital's rule to find

$$\begin{aligned}\lim_{(x,x^2-x)\to(0,0)} \frac{\sin(xy)}{x+y} &= \lim_{x\to 0} \frac{\sin(x^3-x^2)}{x^2} \\ &= \lim_{x\to 0} \frac{\cos(x^3-x^2)(3x-2)}{2} = -\frac{1}{2}.\end{aligned}$$

Therefore $\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{x+y}$ DNE.

c) Along the x-axis

$$\lim_{(x,0)\to(0,0)} \frac{\sin(xy)}{x^2+y^2} = \lim_{x\to 0} \frac{\sin(0)}{x^2} = 0.$$

However, along the line $y = x$ we find

$$\lim_{(x,x)\to(0,0)} \frac{\sin(xy)}{x^2+y^2} = \lim_{x\to 0} \frac{\sin(x^2)}{2x^2} = \frac{1}{2}.$$

Therefore $\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{x^2+y^2}$ DNE.

(3.4) Consider the rational function

$$Q(x,y) = \frac{x^6 - x^5y + xy^5 - y^6}{xy^3 - x^3y}$$

What is the largest subset of the plane $\Omega \subseteq \mathbb{R}^2$ on which we can extend Q to a continuous function? Justify your answer.

Solution (3.4)

$$\begin{aligned}Q(x,y) &= \frac{x^6 - x^5y + xy^5 - y^6}{x^3y - xy^3} \\ &= \frac{(x-y)(x+y)(x^4 - yx^3 + y^2x^2 - y^3x + y^4)}{xy(x-y)(x+y)} \\ &= \frac{x^4 - yx^3 + y^2x^2 - y^3x + y^4}{xy}.\end{aligned}$$

Therefore, by the algebra of continuous functions, Q is a well defined continuous function away from the x and y axes.

If we fix $y \neq 0$, we find that $\lim_{(x,y)\to(0,y)} Q(x,y) = \infty$. Similarly for $x \neq 0$, $\lim_{(x,y)\to(x,0)} Q(x,y) = \infty$. So the only point left to check is the origin. Along the line $y = x^3$, we find that

$$\lim_{x\to 0} Q(x,x^3) = \lim_{x\to 0}(1 - x^2 + x^4 - x^6 + x^8) = 1,$$

but along the line $y = -x^3$

$$\lim_{x\to 0} Q(x,-x^3) = \lim_{x\to 0}(-1 - x^2 - x^4 - x^6 - x^8) = -1,$$

and so $\Omega = \mathbb{R}^2 \setminus (\{x = 0\} \cup \{y = 0\})$ is the plane with the coordinate axes removed.

(3.5) We say a real number is rational if it can be expressed as a ratio of two integers. We denote the collection of all rational numbers as

$$\mathbb{Q} = \{\frac{p}{q} : p, q \in \mathbb{Z},\ q \neq 0\}.$$

At which points in $\mathbb{R}^3$ is the following function continuous?

$$f(x,y,z) := \begin{cases} x + y - z & : x,y,z \in \mathbb{Q} \\ 1 & : \text{ otherwise} \end{cases}.$$

Justify your answer.

Solution (3.5) Consider the hyperplane $P = \{x + y - z = 1\} \subseteq \mathbb{R}^3$, and let $g : \mathbb{R}^3 \to \mathbb{R}$ denote the polynomial (and hence continuous function) $g(x,y,z) = x + y - z$. Fix $\epsilon > 0$ and let $a \in P$. Since g is continuous at a and $g(a) = 1$, there exists $\delta > 0$ such that

$$0 < \|(x,y,z) - a\| < \delta \quad \implies |g(x,y,z) - 1| < \epsilon.$$

More generally, suppose $0 < \|(x,y,z) - a\| < \delta$ holds. If $x, y, z \in \mathbb{Q}$, then $f(x,y,z) = g(x,y,z)$ and we have

$$|f(x,y,z) - f(a)| = |g(x,y,z) - 1| < \epsilon.$$

Conversely, if $x, y, z \notin \mathbb{Q}$, then $f(x,y,z) = 1$ and we vacuously have

$$|f(x,y,z) - f(a)| = |1 - 1| = 0 < \epsilon.$$

Hence f is continuous at every point $a \in P$. Next, we show that f is discontinuous at every point $b \in \mathbb{R}^3 \setminus P$, and thus conclude that f is only continuous at points in P.

To see why f is discontinuous at b, we split our argument into two cases:

Case 1: $f(b) = 1$. We can find a sequence $x_n, y_n, z_n \in \mathbb{Q}$ such that $(x_n, y_n, z_n) \to b$. However,

$$\lim_{n\to\infty} f(x_n,y_n,z_n) = \lim_{n\to\infty} g(x_n,y_n,z_n) = g(b) \neq 1 = f(b),$$

and f is not continuous at b.

Case 2: $f(b) \neq 1$. We find a sequence $(x_n, y_n, z_n) \to b$ with $x_n \notin \mathbb{Q}$. Therefore,

$$\lim_{n\to\infty} f(x_n, y_n, z_n) = \lim_{n\to\infty} 1 \neq f(b),$$

and f is not continuous at b.