Homework 2

Solutions

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(2.1) The angle between a pair of hyperplanes in $\mathbb{R}^n$ is given by the angle formed between a pair of normal vectors to each plane. We make the following convention to choose the normal vectors so that the angle between them is in the range $[0, \frac{\pi}{2}]$. With this convention the angle is well-defined and unique.

Calculate the angle $\theta \in [0, \frac{\pi}{2}]$ between the following pairs of planes:

a)
$$P_1 = \{x + 3y - 2z = 7\}, \quad P_2 = \{-3x + y + 2z = 0\}.$$

b)
$$P_1 = \{(1, 3, 5) + t(-3, 2, 5) + s(0, 0, 1) : t, s \in \mathbb{R}\}$$
$$P_2 = \{t(-1, 1, 0) + s(3, 3, 0) : t, s \in \mathbb{R}\}$$

Solution (2.1)

a) We read off the normal vectors directly $n_1 = (1, 3, -2)$, $n_2 = (3, -1, -2)$. The angle between these vectors is
$$\theta = \arccos\left(\frac{(3 - 3 + 4)}{\sqrt{14}\sqrt{14}}\right) = \arccos\left(\frac{2}{7}\right).$$

b) $n_1 = (-3, 2, 5) \times (0, 0, 1) = (2, 3, 0)$, and $n_2 = (-1, 1, 0) \times (3, 3, 0) = (0, 0, -6)$. Since $n_1 \perp n_2$, we conclude that $\theta = \frac{\pi}{2}$.

(2.2) For any two subsets of $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}^n$, we defined the distance between them to be
$$\inf_{A \in \mathcal{A}, B \in \mathcal{B}} \|\overrightarrow{AB}\|.$$

a) For any two orthogonal vectors $x, y \in \mathbb{R}^n$, show that
$$\|x + y\|^2 = \|x\|^2 + \|y\|^2.$$

b) Let $\mathcal{A} = \{A\} \in \mathbb{R}^3$ be a single point and $\mathcal{B} \subseteq \mathbb{R}^3$ a plane.

Show that there is a unique point $B \in \mathcal{B}$ such that $\overrightarrow{AB}$ is orthogonal to $\mathcal{B}$.

c) Conclude that the distance between $\mathcal{A}$ and $\mathcal{B}$ is $\|\overrightarrow{AB}\|$.

Solution (2.2)

a) Expanding the dot product of $x + y$ with itself, we have

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\underbrace{x \cdot y}_{=0} = \|x\|^2 + \|y\|^2.$$

b) Without loss of generality, we may translate $\mathbb{R}^n$ so that $\mathcal{B}$ passes through the origin.

A generic point $x \in \mathcal{B}$ is given by

$$x = t_1 \cdot v_1 + t_2 \cdot v_2, \quad t_1, t_2 \in \mathbb{R},$$

for linearly independent vectors $v_1, v_2 \in \mathbb{R}^3$. By the exercise from the lectures, we may assume that v_1, v_2 are orthonormal.

Consider the vector from A to a point $C \in \mathcal{B}$

$$\overrightarrow{AC} = t_1 \cdot v_1 + t_2 \cdot v_2 - \overrightarrow{OA}.$$

In order for $\overrightarrow{AC}$ to be orthogonal to $\mathcal{B}$, we require that $\overrightarrow{AC} \cdot v_1 = \overrightarrow{AC} \cdot v_2 = 0$. Since

$$\overrightarrow{AC} \cdot v_1 = t_1 \underbrace{v_1 \cdot v_1}_{=1} + t_2 \underbrace{v_2 \cdot v_1}_{=0} - \overrightarrow{OA} \cdot v_1 = t_1 - \overrightarrow{OA} \cdot v_1,$$

it follows that $t_1 = \overrightarrow{OA} \cdot v_1$. Similarly, $t_2 = \overrightarrow{OA} \cdot v_2$, and therefore $B \in \mathcal{B}$ is the unique point

$$B = \left(\overrightarrow{OA} \cdot v_1\right) v_1 + \left(\overrightarrow{OA} \cdot v_2\right) v_2.$$

c) Using part b), for any $C \in \mathcal{B}$, we can decompose the vector

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC},$$

where $\overrightarrow{AB} \cdot \overrightarrow{BC} = 0$. Therefore by part a)

$$\|\overrightarrow{AB}\| \leq \sqrt{\|\overrightarrow{AB}\|^2 + \|\overrightarrow{BC}\|^2} = \|\overrightarrow{AC}\|, \quad \forall C \in \mathcal{B},$$

and hence $\|\overrightarrow{AB}\|$ is the distance between $\mathcal{A}$ and $\mathcal{B}$ by definition.

(2.3) Let $\gamma : \mathbb{R} \to \mathbb{R}^n$ be a C^1-curve. We define the sphere of radius $r > 0$ at the origin in $\mathbb{R}^n$ by

$$\mathbb{S}_r := \partial B_r(0) = \{x \in \mathbb{R}^n : \|x\| = r\}.$$

Suppose that $\gamma(t)$ is orthogonal to $\gamma'(t)$ for every $t \in \mathbb{R}$. If $\gamma(0)$ is non-zero, show that the curve lies within a sphere. More precisely, show that there exists $r > 0$ such that

$$\gamma(t) \in \mathbb{S}_r, \quad \forall t \in \mathbb{R}.$$

Solution (2.3) Differentiating the length of $\gamma(t)$ in t, we have

$$\frac{d}{dt}\|\gamma(t)\|^2 = 2\gamma(t)\cdot\gamma'(t) = 0,$$

and therefore the length of $\gamma(t)$ is independent of t. Since $r := \|\gamma(0)\| > 0$ by assumption, the curve must always remain in S_r.

(2.4) Let $\gamma : I \to \mathbb{R}^2$ be a curve. Suppose in polar coordinates we have

$$\gamma(t) = (r(t), \theta(t)), \quad \forall t \in I.$$

a) Show that the length of the tangent vector at t in polar coordinates is given by the equation
$$\|\gamma'(t)\| = \sqrt{r'(t)^2 + r^2(t)\theta'(t)^2}.$$

b) For $k > 0$, the logarithmic spiral can be parameterised in polar coordinates as

$$\begin{cases} r(t) = e^{-kt} \\ \theta(t) = t, \end{cases} \qquad \forall t \in [0, \infty).$$

Calculate the arc-length of this curve for each $k > 0$.

Solution (2.4)

a) Recall that in Cartesian coordinates, we have

$$\begin{aligned} x(t) &= r(t)\cos\theta(t), \\ y(t) &= r(t)\sin\theta(t). \end{aligned}$$

Differentiating with respect to t yields:

$$\begin{aligned} x'(t) &= r'(t)\cos\theta(t) - r(t)\theta'(t)\sin\theta(t) \\ y(t) &= r'(t)\sin\theta(t) + r(t)\theta'(t)\cos\theta(t). \end{aligned}$$

Substituting these formula into the equation for the length of the tangent vector in Cartesian coordinates gives

$$\begin{aligned} \|\gamma'(t)\|^2 &= x'(t)^2 + y'(t)^2 \\ &= \left(r'\cos\theta - r\theta'\sin\theta\right)^2 + \left(r'\sin\theta + r\theta'\cos\theta\right)^2 \\ &= r'^2\cos^2\theta + r^2\theta'^2\sin^2\theta - 2rr'\theta'\cos\theta\sin\theta \\ &\quad + r'^2\sin^2\theta + r^2\theta'^2\cos^2\theta + 2rr'\theta'\cos\theta\sin\theta \\ &= (r')^2 + r^2(\theta')^2. \end{aligned}$$

b) Since $r'(t) = -ke^{-kt}$ and $\theta'(t) = 1$, using the formula from part a) we have

$$\begin{aligned} \|\gamma'(t)\| &= \sqrt{r'(t)^2 + r^2(t)\theta'(t)^2} \\ &= \sqrt{k^2 e^{-2kt} + e^{-2kt}} \\ &= e^{-kt}\sqrt{k^2+1}, \end{aligned}$$

and therefore the arc length of the log-spiral is

$$\begin{aligned} S = \int_0^\infty \|\gamma'(t)\|\, dt &= \int_0^\infty e^{-kt}\sqrt{k^2+1}\, dt \\ &= \sqrt{k^2+1}\left(\frac{-e^{-kt}}{k}\right|_0^\infty \\ &= \frac{\sqrt{k^2+1}}{k}. \end{aligned}$$

Remark: As $k \downarrow 0$, the length tends to ∞ as the spiral approaches an infinitely wound unit circle, and as $k \uparrow \infty$, the length tends to 1 as the spiral approaches the line segment joining the point $(1, 0)$ to the origin.