Homework 1

Solutions

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(1.1) Let $u = (1, -2, 1)$, $v = (0, 2, 3)$ and $w = (2, 0, -1)$ be vectors in $\mathbb{R}^3$. Find $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$(1, 1, 1) = \alpha u + \beta v + \gamma w.$$

Solution (1.1) From a simple calculation we find that

$$(1, 1, 1) = 0 \cdot u + \frac{1}{2} \cdot v + \frac{1}{2} \cdot w.$$

(1.2) Let $x, y \in \mathbb{R}^n$.

a) Show that either $(x + y) \cdot x \geq 0$, or $(x + y) \cdot y \geq 0$.

b) Assume now that x and y are non-zero vectors of the same length. Show that, if the vector $x + y$ is non-zero, then it bisects the angle between x and y.

Solution (1.2)

a) It follows from the inequality

$$0 \leq \|x + y\|^2 = (x + y) \cdot x + (x + y) \cdot y,$$

that one of the two values must be non-negative.

b) We first note that

$$(x + y) \cdot x = \|x\| + x \cdot y = \|y\| + x \cdot y = (x + y) \cdot y.$$

Therefore, as $x + y \neq 0$, we have the equality

$$\frac{(x + y) \cdot x}{\|x + y\| \|x\|} = \frac{(x + y) \cdot y}{\|x + y\| \|y\|}.$$

Taking $\arccos$ of both sides, the result follows.

(1.3) For some $m \in \mathbb{N}$, let $x_1, \ldots, x_m \in \mathbb{R}^n$ be unit vectors. The centre of mass of these vectors is defined to be

$$C := \frac{1}{m} \sum_{i=1}^{m} x_i \in \mathbb{R}^n.$$

a) Show that the length of C is less than or equal to 1.

b) When does the length of C equal 1? Justify your answer.

Solution (1.3)

a) By iterating the triangle inequality, it follows that

$$\|C\| \leq \frac{1}{m}\sum_{i=1}^{m}\|x_i\| = \frac{1}{m}\sum_{i=1}^{m} 1 = 1.$$

b) Equality holds in the above inequality iff it holds in each application of the triangle inequality iff $x_1 = \lambda_i x_i$ for some $\lambda_i \geq 0$, for every $i \in \{1, \ldots, m\}$. However, since the vectors are unit vectors, this means that λ_i must always be 1, and so the center of mass has length 1 iff all of the x_i agree.