Midterm Solutions

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(Q1)

a) For a vector $v \in \mathbb{R}^n$, define its length $\|v\|$.

Recall, we call a vector $v \in \mathbb{R}^n$ a unit vector if $\|v\| = 1$.

b) Let $v_1, v_2, v_3 \in \mathbb{R}^3$ be three unit vectors such that $v_1 + v_2 + v_3 = 0$.

Show that $v_1 \cdot (v_2 + v_3) = -1$.

c) Calculate $v_1 \cdot v_2 + v_1 \cdot v_3 + v_2 \cdot v_3$.

d) Can you deduce anything about the triple product of v_1, v_2, v_3? Explain your answer.

Solution (1)

a) The length of v is given by

$$\|v\| = \sqrt{v \cdot v} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}.$$

b) $v_1 \cdot (v_2 + v_3) = v_1 \cdot (-v_1) = -\|v_1\|^2 = -1$

c) By the same reasoning as part b), we see that

$$v_1 \cdot (v_2 + v_3) = v_2 \cdot (v_3 + v_1) = v_3 \cdot (v_1 + v_2) = -1.$$

Therefore, adding these together and using properties of the dot product we find that $-3 = 2(v_1 \cdot v_2 + v_1 \cdot v_3 + v_2 \cdot v_3)$, and hence the sum is equal to $-\frac{3}{2}$.

d) v_1, v_2, v_3 are linearly dependent and so their triple product is zero.

(Q2) In this question, we denote spherical coordinates on $\mathbb{R}^3$ by (ρ, θ, φ).

a) Let $P \in \mathbb{R}^3$ be the point $(\rho, \theta, \varphi) = (4, \frac{\pi}{3}, \frac{\pi}{6})$ in spherical coordinates. Find the Cartesian coordinates (x, y, z) of the point P.

b) For each pair of integers $\alpha, \beta \in \mathbb{Z}$, consider the curve $\gamma_{\alpha,\beta} : [0, 1] \to \mathbb{R}^3$, given in spherical coordinates by the parameterisation

$$\begin{cases} \rho(t) = & 2 + \sin(\frac{\pi}{2}(\alpha + \beta)t), \\ \theta(t) = & 2\pi(\beta + 1)t, \\ \varphi(t) = & \frac{\pi}{2} + \frac{1}{10}\cos(2\pi\alpha t), \end{cases} \qquad \forall t \in [0, 1].$$

For which pairs of integers $\alpha, \beta \in \mathbb{Z}$ is the curve $\gamma_{\alpha,\beta}$ closed?

c) Is the curve $\gamma_{1,3}$ simple? Justify your answer.

d) Calculate the arclength of the curve $\gamma_{0,0}$.

Solution (2)

a) $(x, y, z) = (4\sin(\pi/6)\cos(\pi/3), 4\sin(\pi/6)\sin(\pi/3), 4\cos(\pi/6)) = (1, \sqrt{3}, 2\sqrt{3})$.

b) In order for the curve to be closed we only need to check that $\rho(0) = \rho(1)$, which follows if $\alpha + \beta$ is even.

c) Yes. To see why, suppose $0 \le t_1 < t_2 \le 1$ are such that $\gamma_{1,3}(t_1) = \gamma_{1,3}(t_2)$. Since $\rho(t_1) = \rho(t_2)$ and $\varphi(t_1) = \varphi(t_2)$, it follows that

$$(\sin(2\pi t_1), \cos(2\pi t_1)) = (\sin(2\pi t_2), \cos(2\pi t_2)),$$

which holds iff $t_2 - t_1$ is an integer, and so $t_1 = 0, t_2 = 1$.

d) $\gamma_{0,0}$ is a circle lying in a horizontal plane. Note that

$$x(t) = 2\sin(\frac{\pi}{2} + \frac{1}{10})\cos(2\pi t), \quad y(t) = 2\sin(\frac{\pi}{2} + \frac{1}{10})\sin(2\pi t).$$

Therefore the arclength is $4\pi \sin(\frac{\pi}{2} + \frac{1}{10}) = 4\pi\cos(\frac{1}{10})$.

(Q3) For each of the following sets, state whether the set is:

(i) open, (ii) closed, (iii) bounded, (iv) path connected.

You do not need to justify your reasoning.

a) $S_1 = \{x \in \mathbb{R}^{100} : \|x\| = 1\}$.

b) $S_2 = \{(x, y) \in \mathbb{R}^2 : y > x^2\}$.

c) $S_3 = \mathbb{Q} = \{\frac{p}{q} \in \mathbb{R} : p, q \in \mathbb{Z}, q \neq 0\}$.

d) $S_4 = \{(x, y) \in \mathbb{R}^2 : 0 \le y < \frac{1}{1+x^2}\}$.

Solution (3)

a) Open - No , Closed - Yes, Bounded - Yes , PC - Yes

b) Open - Yes , Closed - No, Bounded - No, PC - Yes

c) Open - No, Closed - No, Bounded - No, PC- No

d) Open - No, Closed - No, Bounded - No, PC - Yes

(Q4) Show that the function $f : \mathbb{R}^4 \to \mathbb{R}$ defined by

$$f(w, x, y, z) = \begin{cases} \frac{\sin(wy+xz)}{wy+xz} & : wy + xz \neq 0 \\ 1 & : wy + xz = 0 \end{cases},$$

is continuous everywhere in $\mathbb{R}^4$.

Solution (4) We begin by defining the function $g : \mathbb{R}^4 \to \mathbb{R}$, $g(w,x,y,z) = wy + xz$. By the algebra of continuous functions, this is continuous. Next, consider the function $h : \mathbb{R} \to \mathbb{R}$ defined by

$$h(t) := \begin{cases} \frac{\sin t}{t} & : t \neq 0 \\ 1 & : t = 0 \end{cases}.$$

By the algebra of continuous functions, h is continuous everywhere apart from the origin. By L'Hôpitals rule

$$\lim_{t \to 0} \frac{\sin t}{t} = \lim_{t \to 0} \frac{\cos t}{1} = 1 = h(0),$$

and therefore h is continuous everywhere. Finally, by the composition of continuous functions, we see that $h \circ g = f$ is continuous everywhere.

(Q5) In this question, consider the function $f : \mathbb{R}^3 \to \mathbb{R}^2$, defined by

$$f(x,y,z) = \begin{pmatrix} f_1(x,y,z) \\ f_2(x,y,z) \end{pmatrix} = \begin{pmatrix} (x-4)^2 + (y-3)^2 + z^2 \\ x - y + 2z \end{pmatrix}.$$

a) For $c \in \mathbb{R}^2$, define what is meant by the level set of the function f at c.

b) Describe the level sets of the first component function

$$f_1(x,y,z) = (x-4)^2 + (y-3)^2 + z^2.$$

c) Describe the level sets of the second component function

$$f_2(x,y,z) = x - y + 2z.$$

d) Describe the level set of f at the point $(1,1) \in \mathbb{R}^2$.

Solution (5)

a) The level set of f at c is all the points

$$L_c := f^{-1}(\{c\}) = \{(x,y,z) \in \mathbb{R}^3 : f(x,y,z) = c\}.$$

b) If $c < 0$, then $L_c = \emptyset$. $L_0 = \{(4,3,0)\}$, and for $c > 0$, L_c is the sphere of radius $\sqrt{c}$ centred at the point $(4,3,0)$.

c) For any c, L_c is the hyperplane with normal vector $(1,-1,2)$ containing the point $(c,0,0)$.

d) The level set of f at $(1,1)$ is the intersection of the level set of f_1 at 1 and the level set of f_2 at 1. In particular, it is the intersection of the sphere centred at $(4,3,0)$ of radius 1, and the hyperplane with normal vector $(1,-1,2)$ passing through the point $(4,3,0)$. Since the centre of the sphere is inside the hyperplane, this intersection is a great circle. That is, the level set is a circle of radius 1 centred at $(4,3,0)$ lying in the hyperplane with normal vector $(1,-1,2)$.

(Q6)

a) State the Squeeze theorem.

b) Suppose $\Omega \subseteq \mathbb{R}^n$ is open, $f, g : \Omega \to \mathbb{R}$, and $a \in \Omega$. If $\lim_{x\to a} f(x) = 0$ and $g(x)$ is bounded for $x \in \Omega$ near a, deduce that $\lim_{x\to a} f(x) \cdot g(x) = 0$.

(A function g is bounded if $|g| \leq C$ holds for some constant $C > 0$.)

c) What does it mean for f to be differentiable at a?

d) Suppose now that $f(x)$ is differentiable at a with $f(a) = 0$ and $\nabla f(a) = 0$. If $g(x)$ is bounded for $x \in \Omega$ near a and $\nabla g(a)$ exists, show that the function $f(x) \cdot g(x)$ is differentiable at a.

Solution (6)

a) Let $f, g, h : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$, $a \in \Omega$, $L \in \mathbb{R}$. If $g(x) \leq f(x) \leq h(x)$ for $x \in \Omega$ near a (i.e $\exists \delta > 0$ such that the inequalities hold for all $x \in \Omega \cap B_\delta(a)$), and $\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L$. Then $\lim_{x\to a} f(x) = L$.

b) For $x \in \Omega$ near a, we find that $|g(x)| \leq C$ for some fixed constant C. In particular, $|f(x)g(x)| \leq C\,|f(x)|$ for $x \in \Omega$ near a. We note that

$$\lim_{x\to a} C\,|f(x)| = C \lim_{x\to a} |f(x)| = C \cdot 0 = 0,$$

and therefore, by the Squeeze theorem, $\lim_{x\to a} f(x) \cdot g(x) = 0$.

c) f is differentiable at a if

- $\nabla f(a)$ exists;
- $\lim_{x\to a} \frac{\varepsilon(x)}{\|x-a\|} = 0$, where

$$\varepsilon(x) = f(x) - f(a) - \nabla f(a)(x-a).$$

d) Since $f(a) = 0$, $\nabla f(a) = 0$, the best affine approx of f at a is the zero function. Since f is differentiable at a, we conclude that $\lim_{x\to a} \frac{f(x)}{\|(x-a)\|} = 0$. By the product rule, we have that

$$\nabla(fg)(a) = f(a)\nabla g(a) + g(a)\nabla f(a) = 0,$$

and therefore the best affine approximation for the product fg is also the zero function. Thus, we need to show that $\lim_{x\to a} \frac{f(x)g(x)}{\|(x-a)\|} = 0$. But since $g(x)$ is bounded for x near a, we can apply part b).