Makeup Midterm Solutions

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(Q1) For this question, suppose $v_1, v_2, v_3 \in \mathbb{R}^3 \setminus \{0\}$ are three non-zero vectors which sum to zero

$$v_1 + v_2 + v_3 = 0.$$

Assume also that these three vectors are pairwise non-parallel, i.e v_i is not parallel to v_j for any $i, j \in \{1, 2, 3\}$.

a) State the triangle inequality for vectors in $\mathbb{R}^n$, making sure to mention when equality holds.

b) Show that

$$\|v_1\| < \|v_2\| + \|v_3\|.$$

c) Consider the function $f : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$f(x, y, z) = \begin{pmatrix} f_1(x, y, z) \\ f_2(x, y, z) \\ f_3(x, y, z) \end{pmatrix} = \begin{pmatrix} v_1 \cdot (x, y, z) \\ v_2 \cdot (x, y, z) \\ v_3 \cdot (x, y, z) \end{pmatrix}.$$

Describe geometrically the level sets of the first component function $f_1 : \mathbb{R}^3 \to \mathbb{R}$.

d) For which vectors $c = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \in \mathbb{R}^3$ is the level set of f at c non-empty?

When the level set is non-empty, give a geometric description of the level set.

Solution (1)

a) For any pair of vectors $x, y \in \mathbb{R}^n$, we have

$$\|x + y\| \le \|x\| + \|y\|,$$

with equality iff $x = \lambda y$ or $y = \lambda x$ for some $\lambda \ge 0$.

b)

$$\|v_1\| = \|-(v_2 + v_3)\| = \|v_2 + v_3\| \le \|v_2\| + \|v_3\|.$$

Since v_2 is not parallel to v_3, we do not have equality in our application of the triangle inequality, and hence

$$\|v_1\| < \|v_2\| + \|v_3\|.$$

c) For any real number $c \in \mathbb{R}$, the level set of f_1 at c is a hyperplane with normal vector v_1. As we vary c, we translate the hyperplane. Letting c run through all the values of $\mathbb{R}$, we cover all of $\mathbb{R}^3$ with parallel hyperplanes.

d) We note that

$$(f_1 + f_2 + f_3)(x, y, z) = (v_1 + v_2 + v_3) \cdot (x, y, z) = 0.$$

Therefore, the level set at c is non-empty iff $c_1 + c_2 + c_3 = 0$. When the level set is non-empty, it is the intersection of 3 hyperplanes. Since the vectors are pairwise non-parallel and linearly dependent, the level set is a line with direction $v_1 \times v_2$. Varying c within the set of vectors such that $c_1 + c_2 + c_3 = 0$ translates the line. Letting c run through all such values, we cover all of $\mathbb{R}^3$ with parallel lines.

(Q2)

a) State the Cauchy-Schwarz inequality.

b) For any four real numbers $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}$, show that the following inequality holds

$$\frac{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4}{2} \leq \left(\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2\right)^{\frac{1}{2}}.$$

c) Consider the curve $\gamma : [0, 1] \to \mathbb{R}^4$ given by

$$\gamma(t) = \begin{pmatrix} t + \sin^2(t) + \cos^3(t) - 2\sin(t)\cos^2(t) \\ -3t + cos^2(t) + \cos^5(t)\sin^3(t) + 2\sin(t)\cos^2(t) \\ 2t - \cos^3(t) - \sin^2(t) \\ 2t - \cos^5(t)\sin^3(t) - \cos^2(t) \end{pmatrix}, \quad \forall t \in [0, 1].$$

If S denotes the arclength of γ, using your result from part b) or otherwise, show that $S \geq 1$.

Solution (2)

a) For any pair of vectors $x, y \in \mathbb{R}^n$, we have

$$|x \cdot y| \leq \|x\|\|y\|,$$

with equality iff x and y are parallel.

b) Let $x = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{R}^4$ and $y = (1, 1, 1, 1) \in \mathbb{R}^4$. Applying Cauchy-Schwarz we have that

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = x \cdot y \leq \|x\|\|y\| = \left(\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2\right)^{\frac{1}{2}} (1+1+1+1)^{\frac{1}{2}}.$$

Dividing through by 2 gives the desired inequality.

c) Since the arclength of $\gamma(t)$ is given by

$$\int_0^1 \|\gamma'(t)\| dt,$$

if we can show that the speed $\|\gamma'(t)\| \geq 1$ for all $t \in [0,1]$, then we are done.

Since $(\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4)(t) = 2t$, we apply part b) to find that

$$\begin{aligned}
\|\gamma'(t)\| &= \left(\gamma_1'(t)^2 + \gamma_2'(t)^2 + \gamma_3'(t)^2 + \gamma_4'(t)^2\right)^{\frac{1}{2}} \\
&\geq \frac{1}{2}\left(\gamma_1'(t) + \gamma_2'(t) + \gamma_3'(t) + \gamma_4'(t)\right) \\
&= \frac{1}{2}\left(\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4\right)'(t) \\
&= \frac{1}{2}\frac{d}{dt}(2t) = 1.
\end{aligned}$$

(Q3) In this question, we denote cylindrical coordinates on $\mathbb{R}^3$ by (r, θ, z).

a) Let $P \in \mathbb{R}^3$ be the point $(r, \theta, z) = (4\sqrt{2}, \frac{\pi}{4}, 6)$ in cylindrical coordinates. Find the Cartesian coordinates (x, y, z) of the point P.

b) For each pair of integers $\alpha, \beta \in \mathbb{Z}$, consider the curve $\gamma_{\alpha,\beta} : [0,1] \to \mathbb{R}^3$, given in cylindrical coordinates by the parameterisation

$$\begin{cases} r(t) = & 10 + \sin(\frac{\pi}{2}(\alpha+\beta)t), \\ \theta(t) = & \pi\beta t, \\ z(t) = & 5\cos(2\pi\alpha t), \end{cases} \qquad \forall t \in [0,1].$$

For which pairs of integers $\alpha, \beta \in \mathbb{Z}$ is the curve $\gamma_{\alpha,\beta}$ closed?

c) Is the curve $\gamma_{2024,2}$ simple? Justify your answer.

d) Is the curve $\gamma_{4,4}$ simple? Justify your answer.

Solution (3)

a) $(x, y, z) = (4\sqrt{2}\cos\pi/4, 4\sqrt{2}\sin\pi/4, 6) = (4, 4, 6)$.

b) $\gamma_{\alpha,\beta}$ is closed if $\gamma_{\alpha,\beta}(0) = \gamma_{\alpha,\beta}(1)$, which is true provided

$$\begin{aligned}
&\sin(\frac{\pi}{2}(\alpha+\beta)) = 0 \iff \alpha+\beta \text{ is even.} \\
&\pi\beta \in 2\pi\mathbb{Z} \iff \beta \text{ is even.} \\
&\cos(2\pi\alpha) = 1, \text{ which is always true.}
\end{aligned}$$

Thus $\gamma_{\alpha,\beta}$ is closed iff α and β are both even.

c) Yes. $\theta(t) - \theta(s) \in 2\pi\mathbb{Z}$ can only happen when $\{t, s\} = \{0, 1\}$.

d) No. In Cartesian coordinates we have

$$\gamma_{4,4}(0) = (10, 0, 5) = \gamma_{4,4}(\frac{1}{2}).$$

(Q4) For each of the following sets, state whether the set is:

(i) open, (ii) closed, (iii) bounded, (iv) path connected.

You do not need to justify your reasoning.

a) $S_1 = \mathbb{R}^n$ (as a subset of itself).

b) $S_2 = \{(x,y,z) \in \mathbb{R}^3 : x + 4y - 2z = 6\}$.

c) $S_3 = \{(x,y) \in \mathbb{R}^2 : |x| + |y| \leq 10\}$.

d) $S_4 = \{(x,y,z) \in \mathbb{R}^3 : 1 < e^{x^2+y^2} < 2\}$.

Solution (4)

a) S_1 is open, closed and path connected, but not bounded.

b) S_2 is closed and path connected, but not open or bounded.

c) S_3 is closed, bounded and path connected, but not open.

d) S_4 is open and path connected, but not closed or bounded.

(Q5) Show that the function $F : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$F(x,y,z) = \begin{cases} \frac{\exp(\frac{1}{yz-x^3})}{(x^3-yz)} & : x^3 - yz > 0 \\ 0 & : x^3 - yz \leq 0 \end{cases},$$

is continuous everywhere in $\mathbb{R}^3$.

Solution (5) The function $g : \mathbb{R}^3 \to \mathbb{R}$ defined by $g(x,y,z) = x^3 - yz$ is a polynomial and hence continuous. Next, consider the function $h : \mathbb{R} \to \mathbb{R}$ defined by

$$h(t) = \begin{cases} \frac{e^{-t^{-1}}}{t} & : t > 0 \\ 0 & : t \leq 0 \end{cases}.$$

Note that by the algebra of continuous functions h is continuous away from zero. By L'Hôpital's rule

$$\lim_{t \downarrow 0} h(t) = \lim_{t \downarrow 0} \frac{t^{-1}}{e^{t^{-1}}} = \lim_{t \downarrow 0} \frac{-t^{-2}}{-t^{-2} e^{t^{-1}}} = \lim_{t \downarrow 0} \frac{1}{e^{t^{-1}}} = 0 = h(0).$$

Therefore, h is continuous everywhere on $\mathbb{R}$. By the composition of continuous functions $f = h \circ g$ is continuous on $\mathbb{R}^3$.

(Q6) Suppose $\Omega \subseteq \mathbb{R}^n$ is open, $f, g : \Omega \to \mathbb{R}$, and $a \in \Omega$. Define a new function $h : \Omega \to \mathbb{R}$ given by the product of f and g,

$$h(x) := f(x)g(x), \quad \forall x \in \Omega.$$

a) Define what it means for the function f to be differentiable at a

b) Suppose f is differentiable at a with $f(a) = 0$ and $\nabla f(a) = 0$. If g is continuous at a, show that the function $h(x)$ is differentiable at a.

c) If we remove the hypothesis that $f(a) = 0$, is the result from part c) still true? That is, if f is differentiable at a with $\nabla f(a) = 0$, and g is continuous at a, is the function $h(x)$ necessarily differentiable at a? Either provide a proof or a counterexample.

Solution (6)

a) f is differentiable at a if $\nabla f(a)$ exists and the error function

$$\varepsilon(x) = f(x) - f(a) - \nabla f(a)(x - a),$$

satisfies $\lim_{x \to a} \frac{\varepsilon(x)}{\|x-a\|} = 0$.

b) Since f is differentiable at a with $f(a) = 0$ and $\nabla f(a) = 0$, the definition of differentiability tells us that $\varepsilon = f$, and so $\lim_{x \to a} \frac{f(x)}{\|x-a\|} = 0$.

Since g is continuous at a, by taking $\epsilon = 1$ in the definition, we conclude that $|g(x)| \leq C = (|g(a)| + 1)$ for x near a.

Since

$$\lim_{s \to 0} \frac{f(a + se_i)}{s} = \lim_{s \to 0} \frac{f(a + se_i) - f(a)}{s} = \frac{\partial f}{\partial x_i}(a) = 0,$$

we conclude that

$$\lim_{s \to 0} \frac{C\,|f(a + se_i)|}{|s|} = 0.$$

Then, as

$$|h(x)| = |f(x)|\,|g(x)| \leq C\,|f(x)|\,,$$

for x near a, by the Squeeze Theorem,

$$\lim_{s \to 0} \frac{h(a + se_i) - h(a)}{s} = \lim_{s \to 0} \frac{h(a + se_i)}{s} = \lim_{s \to 0} \frac{C\,|f(a + se_i)|}{|s|} = 0.$$

Therefore, $\nabla h(a)$ exists and equals zero. Finally, since $h(a) = 0$ and $\nabla h(a) = 0$, it follows that

$$\begin{aligned}
\lim_{x \to a} \frac{h(x)}{\|x - a\|} &= \lim_{x \to a} \frac{f(x)}{\|x - a\|} \cdot g(x) \\
&= \lim_{x \to a} \frac{f(x)}{\|x - a\|} \cdot \lim_{x \to a} g(x) \\
&= 0 \cdot g(a) = 0,
\end{aligned}$$

and h is differentiable at a.

c) No. As a counterexmaple take $f \equiv 1$ and $g(x) = |x|$. Then f is differentiable at 0 with $f'(0) = 0$, and g is continuous at 0, but $h(x) = |x|$ is not differentiable at $x = 0$.