Homework 6

Please submit, through Blackboard, solutions to *all* of the following problems. The deadline for submissions is 18:00 on Tuesday 25th June 2024. Please let me know if any of the problems are unclear or have typos.

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(6.1) In this question, consider the following function $f : \mathbb{R} \to [0, \infty)$

$$f(t) = \begin{cases} \exp(-\frac{1}{t}) & : t > 0 \\ 0 & : t \leq 0 \end{cases}.$$

a) Show that $f(t)$ is a smooth function.

b) Calculate the k^{th}-order Taylor polynomial of $f(t)$ at $t = 0$ for any $k \in \mathbb{N}$.

c) Define the function

$$F(x) = \frac{f(2 - \|x\|)}{f(2 - \|x\|) + f(\|x\| - 1)}, \quad \forall x \in \mathbb{R}^n.$$

Show that F is a smooth function on $\mathbb{R}^n$ with

$$0 \leq F(x) \leq 1, \quad \forall x \in \mathbb{R}^n.$$

Moreover, show that $F(x) = 1$ if $\|x\| \leq 1$, and $F(x) = 0$ if $\|x\| \geq 2$.

(6.2)

a) Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a continuous function such that $\lim_{x\to\infty} f(x) = \infty$. That is

$$\forall C \in \mathbb{R},\ \exists R > 0 \quad \text{such that} \quad \|x\| \geq R \implies f(x) > C.$$

Show that f attains a global minimum on $\mathbb{R}^n$.

b) Suppose $g : \mathbb{R}^n \to (0, \infty)$ is a positive continuous function such that $\lim_{x\to\infty} g(x) = 0$. That is

$$\forall \epsilon > 0,\ \exists R > 0 \quad \text{such that} \quad \|x\| \geq R \implies g(x) < \epsilon.$$

Show that g attains a global maximum on $\mathbb{R}^n$.

c) Does the function $g : \mathbb{R}^n \to (0, \infty)$ from part b) necessarily attain a global minimum? Justify your answer.

d) Find the global maximum of the function

$$h(x, y) = \frac{1 + |x| + |y|}{1 + x^2 + y^2}, \quad \forall (x, y) \in \mathbb{R}^2.$$

(6.3) Consider the function $F : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$F(x, y) = \sin(x)\sin(y), \quad \forall (x, y) \in \mathbb{R}^2.$$

a) Find and classify the critical points of F.

b) At each critical point, find the 2^{nd}-order Taylor polynomial P_2.