Homework 5

Please submit, through Blackboard, solutions to *all* of the following problems. The deadline for submissions is 18:00 on Monday 17th June 2024. Please let me know if any of the problems are unclear or have typos.

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(5.1) In each of the following examples, find the partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ at the point (x_0, y_0, z_0):

a)
$$f(t) = e^t, \quad (x_0, y_0, z_0) = (0, 0, 0), \quad t = (x + z)e^y.$$

b)
$$f(u, v) = \arctan(uv), \quad (x_0, y_0, z_0) = (0, 0, 0),$$
$$u = x \sin y \sin z + x \cos z, \quad v = x \cos y \sin z.$$

c)
$$f(a, b, c, d) = a^2 b + c^2 d - e^{ac} \log(bd), \quad (x_0, y_0, z_0) = (1, 1, 1),$$
$$a = x + y + z, \quad b = x^2, \quad c = y^2, \quad d = z^2.$$

(5.2) Consider the curve determined by the equation

$$xe^y + \sin(xy) + y = \log 2.$$

a) At which points does the curve intersect the horizontal line $y = \log 2$.

b) Assume that y is locally given as a function of x around each of the points found in part a). Find the value of $\frac{dy}{dx}$ at these points.

(5.3) Suppose $f : \Omega_1 \subseteq \mathbb{R}^n \to \Omega_2 \subseteq \mathbb{R}^n$ is a bijection, with inverse $f^{-1} : \Omega_2 \to \Omega_1$.

a) Let $I : \mathbb{R}^n \to \mathbb{R}^n$ be the identity map. Calculate its Jacobian matrix at any point of $\mathbb{R}^n$.

b) Suppose f is differentiable at $a \in \Omega_1$ and f^{-1} is differentiable at $f(a) \in \Omega_2$. Express the Jacobian matrix of f^{-1} at $f(a)$ in terms of $Df(a)$.

c) Consider the particular example of the smooth bijection $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$. Using part b), calculate the derivative of $f^{-1}(y)$ at $y = 1$.

d) Explain why we cannot use part b) when $x = 0$.

(5.4) Consider a curve $\gamma : I \to \mathbb{R}^3$ parameterised in Spherical coordinates

$$\gamma(t) = (\rho(t), \varphi(t), \theta(t)), \quad t \in I.$$

a) Using the formulas for the Cartesian coordinates

$$\begin{aligned} x &= \rho \sin\varphi \cos\theta, \\ y &= \rho \sin\varphi \sin\theta, \\ z &= \rho \cos\varphi, \end{aligned}$$

find equations for $x'(t)$, $y'(t)$, and $z'(t)$ in terms of $\rho'(t)$, $\varphi'(t)$, and $\theta'(t)$.

b) Substituting your answers from a) into the formula

$$\|\gamma'(t)\| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2},$$

find an equation for the speed of the curve in spherical coordinates.

c) Using the formula from b) or otherwise, calculate the arclength of the curve

$$\begin{cases} \rho(t) = & e^t, \\ \theta(t) = & \log(\tan t), \\ \varphi(t) = & 2t, \end{cases} \quad \forall t \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right).$$