Homework 4

Please submit, through Blackboard, solutions to *all* of the following problems. The deadline for submissions is 18:00 on Monday 10th June 2024. Please let me know if any of the problems are unclear or have typos.

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(4.1) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} \frac{x^3}{x^2+y^2} & : (x, y) \neq (0, 0) \\ 0 & : (x, y) = (0, 0) \end{cases}.$$

a) Calculate the directional derivative $D_u f(0, 0)$ for the vector $u = (\cos\theta, \sin\theta)$.

b) Calculate the gradient $\nabla f(0, 0)$.

c) Is the function f differentiable at the origin? Justify your answer.

(4.2) Consider the family of functions

$$f_\alpha(x, y, z) = e^x + y\cos z + \alpha(x^3 - y + e^{-z}\sin x), \quad \forall \alpha \in \mathbb{R}.$$

For which values of α does the function increase most rapidly in a direction parallel to the x-axis at the origin? Justify your answer.

(4.3) Let $\Omega \subseteq \mathbb{R}^n$ open, $f : \Omega \to \mathbb{R}$ and $a \in \Omega$. We assume that there exists some affine function $L : \mathbb{R}^n \to \mathbb{R}$, defined by

$$L(x) = \lambda + \alpha \cdot (x - a), \quad (\lambda \in \mathbb{R},\ \alpha \in \mathbb{R}^n)$$

such that the error function $\varepsilon(x) = f(x) - L(x)$ satisfies

$$\lim_{x \to a} \frac{\varepsilon(x)}{\|x - a\|} = 0.$$

a) Show that $\lim\limits_{x \to a} f(x)$ exists and equals λ.

b) Suppose also that all the partial derivatives $\frac{\partial f}{\partial x_i}(a)$ (for $1 \leq i \leq n$) exist.

Show that f is continuous at a.

c) Conclude that f is differentiable at a.

(4.4) An important partial differential equation that describes the distribution of heat in a region at time t can be represented by the one-dimensional heat equation

$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2}.$$

For constants $\alpha, \beta \in \mathbb{R}$, consider the function

$$u(x,t) := \sin(\alpha x)e^{-\beta t}.$$

Find a relationship between the constants α and β for this function to be a solution to the one-dimensional heat equation.