Homework 3

Please submit, through Blackboard, solutions to *all* of the following problems. The deadline for submissions is 18:00 on Monday 3rd June 2024. Please let me know if any of the problems are unclear or have typos.

. . . .

(3.1)

a) Let $x_0 \in \mathbb{R}^n$ and $r > 0$. Show that the set

$$B_r(x_0) = \{x \in \mathbb{R}^n : \|x - x_0\| < r\},$$

is an open set.

This justifies naming $B_r(x_0)$ an open ball.

b) Show that, for a pair of open sets X_1, X_2, their union $X_1 \cup X_2$ is also an open set.

c) Show that, for a pair of open sets X_1, X_2, their intersection $X_1 \cap X_2$ is also an open set.

d) Suppose now we have a (not necessarily finite) collection of open sets

$$\{X_\alpha : \alpha \in \mathcal{A}\}.$$

Is their union $U := \bigcup_{\alpha \in \mathcal{A}} X_\alpha$ necessarily open? Justify your answer.

e) Is their intersection $C := \bigcap_{\alpha \in \mathcal{A}} X_\alpha$ necessarily open? Justify your answer.

(3.2) Let $A \subseteq \mathbb{R}^n$ and consider the set of vectors whose distance to A is zero:

$$\{x \in \mathbb{R}^n : \forall \epsilon > 0, \exists a \in A \text{ such that } \|x - a\| < \epsilon\}.$$

Show that this set is the same as the closure of A:

$$\overline{A} = \text{Int}(A) \cup \partial A.$$

(3.3) For the following examples, does the limit exist, and if so, what is its value?

a) $\lim\limits_{(x,y)\to(0,0)} \frac{\sin(xy)}{xy}$

b) $\lim\limits_{(x,y)\to(0,0)} \frac{\sin(xy)}{x+y}$

c) $\lim\limits_{(x,y)\to(0,0)} \frac{\sin(xy)}{x^2+y^2}$

(3.4) Consider the rational function

$$Q(x,y) = \frac{x^6 - x^5y + xy^5 - y^6}{xy^3 - x^3y}$$

What is the largest subset of the plane $\Omega \subseteq \mathbb{R}^2$ on which we can extend Q to a continuous function? Justify your answer.

(3.5) We say a real number is rational if it can be expressed as a ratio of two integers. We denote the collection of all rational numbers as

$$\mathbb{Q} = \{\frac{p}{q} : p, q \in \mathbb{Z},\ q \neq 0\}.$$

At which points in $\mathbb{R}^3$ is the following function continuous?

$$f(x,y,z) := \begin{cases} x + y - z & : x, y, z \in \mathbb{Q} \\ 1 & : \text{ otherwise} \end{cases}.$$

Justify your answer.