Homework 2

Please submit, through Blackboard, solutions to *all* of the following problems. The deadline for submissions is 18:00 on Monday 27th May 2024. Please let me know if any of the problems are unclear or have typos.

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(2.1) The angle between a pair of hyperplanes in $\mathbb{R}^n$ is given by the angle formed between a pair of normal vectors to each plane. We use the convention that our choice of normal vectors form an angle between them in the range $[0, \frac{\pi}{2}]$. With this convention the angle is well-defined and unique.

Calculate the angle $\theta \in [0, \frac{\pi}{2}]$ between the following pairs of planes:

a)
$$P_1 = \{x + 3y - 2z = 7\}, \quad P_2 = \{-3x + y + 2z = 0\}.$$

b)
$$P_1 = \{(1, 3, 5) + t(-3, 2, 5) + s(0, 0, 1) : t, s \in \mathbb{R}\},$$
$$P_2 = \{t(-1, 1, 0) + s(3, 3, 0) : t, s \in \mathbb{R}\}.$$

(2.2) For any two subsets of $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}^n$, we defined the distance between them to be

$$\inf_{A \in \mathcal{A}, B \in \mathcal{B}} \|\overrightarrow{AB}\|.$$

a) For any two orthogonal vectors $x, y \in \mathbb{R}^n$, show that

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2.$$

b) Let $\mathcal{A} = \{A\} \in \mathbb{R}^3$ be a single point and $\mathcal{B} \subseteq \mathbb{R}^3$ a plane.

Show that there is a unique point $B \in \mathcal{B}$ such that $\overrightarrow{AB}$ is orthogonal to $\mathcal{B}$.

c) Conclude that the distance between $\mathcal{A}$ and $\mathcal{B}$ is $\|\overrightarrow{AB}\|$.

(2.3) Let $\gamma : \mathbb{R} \to \mathbb{R}^n$ be a C^1-curve. We define the sphere of radius $r > 0$ at the origin in $\mathbb{R}^n$ by

$$\mathbb{S}_r := \partial B_r(0) = \{x \in \mathbb{R}^n : \|x\| = r\}.$$

Suppose that $\gamma(t)$ is orthogonal to $\gamma'(t)$ for every $t \in \mathbb{R}$. If $\gamma(0)$ is non-zero, show that the curve lies within a sphere. More precisely, show that there exists $r > 0$ such that

$$\gamma(t) \in \mathbb{S}_r, \quad \forall t \in \mathbb{R}.$$

(2.4) Let $\gamma : I \to \mathbb{R}^2$ be a curve. Suppose in polar coordinates we have

$$\gamma(t) = (r(t), \theta(t)), \quad \forall t \in I.$$

a) Show that the speed of the curve in polar coordinates is given by the equation

$$\|\gamma'(t)\| = \sqrt{r'(t)^2 + r^2(t)\theta'(t)^2}.$$

b) For $k > 0$, the logarithmic spiral can be parameterised in polar coordinates as

$$\begin{cases} r(t) = e^{-kt} \\ \theta(t) = t, \end{cases} \qquad \forall t \in [0, \infty).$$

Calculate the arc-length of this curve for each $k > 0$.