CHINESE UNIVERSITY OF HONG KONG

MATH2010E
Advanced Calculus I

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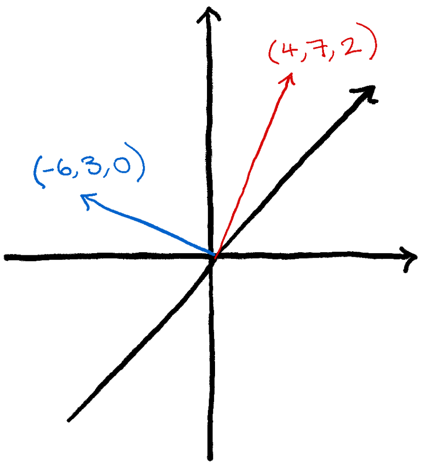
1 Week 1

1.1 Euclidean space $\mathbb{R}^n$

In this course, we are concerned with the analysis of functions between finite dimensional vector spaces over the real numbers $\mathbb{R}$. For some fixed dimension $n \in \mathbb{N}$, up to isomorphism such a vector space is n-dimensional Euclidean space $\mathbb{R}^n$.

$$\mathbb{R}^n = \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{n-times} = \{(x_1, \ldots, x_n) : x_i \in \mathbb{R}, \quad \forall i \in \{1, \ldots, n\}\}.$$



Notation: In these printed notes we will use regular lettering $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ to denote a vector, with subscripts denoting the component of such a vector in Cartesian coordinates. In some cases, we will also use capital letters $A, B, C, \ldots$ to represents points within $\mathbb{R}^n$, and use $\overrightarrow{AB}$ to denote the vector starting from point A and ending at point B.

Basic Operations on Vectors

Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ be a pair of vectors and $\lambda \in \mathbb{R}$ a scalar.

- Equality: $x = y \iff x_i = y_i$, for all $i \in \{1, \ldots, n\}$.

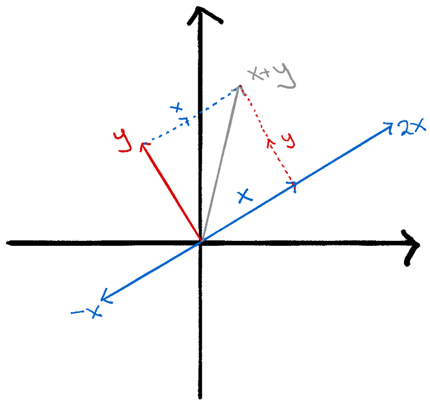
  'Two vectors are the same if and only if all of their components agree.'

- Addition: $x + y = (x_1 + y_1, \ldots, x_n + y_n)$.

  'Vectors are summed component wise.'

- Scalar multiplication: $\lambda x = (\lambda x_1, \ldots, \lambda x_n)$.

  'Scaling a vector scales all of its components by the same amount.'



Length and the Dot Product

As well as the usual vector space structure of $\mathbb{R}^n$, we also equip it with its natural inner product, known as the *dot product*.

Definition 1. *For any pair of vectors $x, y \in \mathbb{R}^n$ we define their dot product to be*

$$x \cdot y := \sum_{i=1}^{n} x_i y_i \in \mathbb{R}.$$

For any vector $x \in \mathbb{R}^n$ we define its (Euclidean) length $\|x\|$ via the equation

$$\|x\|^2 := x \cdot x = x_1^2 + \cdots + x_n^2.$$

We note that the dot product is a map $\mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$. Moreover $0 \cdot x = 0$, for every $x \in \mathbb{R}^n$.

Remark. *In dimension $n \leq 3$, $\|x\|$ agrees with the usual notion of length you calculate by applying the Pythagorean theorem.*

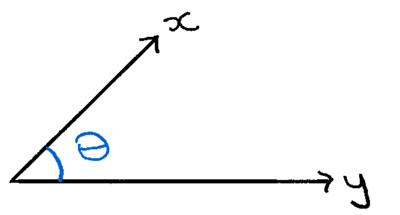
Lemma 1. *Let $x, y, z \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. Then the dot product enjoys the following properties.*

a) $x \cdot y = y \cdot x$.

b) $(x + y) \cdot z = x \cdot z + y \cdot z$.

c) $(\lambda x) \cdot y = \lambda(x \cdot y) = x \cdot (\lambda y)$.

d) $x \cdot x \geq 0$, *with equality iff* $x = 0 \in \mathbb{R}^n$.



e) If $x \neq 0$ and $y \neq 0$, then $x \cdot y = \|x\|\|y\| \cos\theta$,

where θ is the size of the angle formed between the vectors x and y.

Proof. Properties a)-d) follow trivialy from the definitions and are left as an exercise. To show e) we first consider the case where $\theta \in [0, \frac{\pi}{2}]$. By the definition of the dot product, we have

$$\begin{aligned}\|y - x\|^2 &= (y - x) \cdot (y - x) \\ &= y \cdot y - x \cdot y - y \cdot x + x \cdot x \\ &= \|y\|^2 + \|x\|^2 - 2x \cdot y.\end{aligned}$$

Alternatively, by the Pythagorean theorem

$$\begin{aligned}\|y - x\|^2 &= (\|x\| - \|y\| \cos\theta)^2 + (\|y\| \sin\theta)^2 \\ &= \|y\|^2 + \|x\|^2 - 2\|x\|\|y\| \cos\theta.\end{aligned}$$

Comparing the two gives the result for $\theta \in [0, \frac{\pi}{2}]$. Finally, if $\theta \in (\frac{\pi}{2}, \pi]$, we consider instead the vectors x and $-y$, which now form an angle of size $\pi - \theta \in [0, \frac{\pi}{2})$. We may then apply the above result to conclude that

$$x \cdot (-y) = \|x\|\|-y\| \cos(\pi - \theta).$$

Since $x \cdot (-y) = -(x \cdot y)$, $\|-y\| = \|y\|$, and $\cos(\pi - \theta) = -\cos(\theta)$, the result follows. □

Remark. *For any two vectors $x, y \in \mathbb{R}^n$ which form an angle of size θ,*

$$x \text{ and } y \text{ are orthogonal} \iff \theta = \frac{\pi}{2} \iff \cos(\theta) = 0 \iff x \cdot y = 0.$$

The following is a simple lemma which states that the diagonals of a parallelogram are orthogonal to each other if and only if the parallelogram is in fact a rhombus.

Lemma 2. *For any two vectors $x, y \in \mathbb{R}^n$, $x + y$ is orthogonal to $x - y$ if and only if x and y have the same length.*

Proof. From the previous remark, $x + y$ and $x - y$ are orthogonal iff $(x + y) \cdot (x - y) = 0$, but a direct calculation leads to

$$(x + y) \cdot (x - y) = x \cdot x + x \cdot y - x \cdot y - y \cdot y = \|x\|^2 - \|y\|^2,$$

from which the result follows immediately. □

As an application of the previous lemma, we give a simple proof that for any triangle lying on a circle with one edge given by a diameter, the opposite angle is $\frac{\pi}{2}$.

If $x = \overrightarrow{OC}$ and $y = \overrightarrow{AO}$, then $\overrightarrow{AC} = x + y$ and $\overrightarrow{BC} = x - y$. Since the length of x and y are the same (the radius of the circle) by the previous lemma, these two vectors are orthogonal.

Cauchy-Schwarz and the Triangle Inequality.

The following is an important inequality which bounds the size of the dot product by the product of the lengths.

Lemma 3 (Cauchy-Schwarz inequality)**.** *For any $x, y \in \mathbb{R}^n$,*

$$|x \cdot y| \leq \|x\|\|y\|. \tag{1.1}$$

Moreover, equality holds if and only if x and y are parallel. That is, $x = \lambda y$ or $y = \lambda x$ for some $\lambda \in \mathbb{R}$.

Remark. *For $n \leq 3$, this follows immediately from the equality $x \cdot y = \|x\|\|y\| \cos\theta$.*

Proof. If $y = 0 \in \mathbb{R}^n$, then the result is trivially true with $y = 0 \cdot x$. Therefore, we may assume throughout the proof that $y \neq 0$. For any $\lambda \in \mathbb{R}$, we consider the quantity

$$0 \leq \|x - \lambda y\|^2 := (x - \lambda y) \cdot (x - \lambda y) = \|x\|^2 - 2\lambda x \cdot y + \lambda^2 \|y\|^2. \tag{1.2}$$

As $y \neq 0$, we may choose $\lambda := \frac{x \cdot y}{\|y\|^2} \in \mathbb{R}$ in (1.2), which simplifies to

$$\|x\|^2 - \frac{|x \cdot y|}{\|y\|^2} \geq 0.$$

Rearranging, we can conclude (1.1) holds. Moreover, equality in (1.1) holds iff $\|x - \lambda y\| = 0$ for our choice of λ, which happens iff x and y are parallel. □

Remark. *Let $x, y \in \mathbb{R}^n$ be non-zero vectors. For $n \leq 3$, we proved that*

$$x \cdot y = \|x\|\|y\| \cos\theta.$$

Rearranging, we have

$$\theta = \arccos\left(\frac{x \cdot y}{\|x\|\|y\|}\right) \in [0, \pi].$$

For any $n \in \mathbb{N}$, Cauchy-Schwarz implies that

$$-1 \leq \frac{x \cdot y}{\|x\|\|y\|} \leq 1.$$

Thus, we can define

$$\theta = \arccos\left(\frac{x \cdot y}{\|x\|\|y\|}\right) \in [0, \pi],$$

to be the angle between x and y in all dimensions.

Example 4. For the pair of vectors $x = (1, 5, 2, 3, 4), y = (3, 3, 1, 1, 0) \in \mathbb{R}^5$, we have

$$x \cdot x = 55, \quad y \cdot y = 20, \quad x \cdot y = 23.$$

Therefore, the angle between them is $\theta = \arccos\left(\frac{2.3}{\sqrt{11}}\right)$.

Just like for the absolute value in $\mathbb{R}$, the triangle inequality holds for the length of vectors in $\mathbb{R}^n$ as well.

Lemma 5. *For any $x, y \in \mathbb{R}^n$,*

$$\|x + y\| \leq \|x\| + \|y\|. \tag{1.3}$$

Moreover, equality holds if and only if $x = \lambda y$ or $y = \lambda x$ for some $\lambda \geq 0$.

Proof. By expanding the dot product and using the Cauchy-Schwarz inequality

$$\begin{aligned}
\|x + y\|^2 &= \|x\|^2 + 2x \cdot y + \|y\|^2 \\
&\leq \|x\|^2 + 2\,|x \cdot y| + \|y\|^2 \\
&\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \\
&= (\|x\| + \|y\|)^2,
\end{aligned}$$

which is precisely (1.3). Moreover, we have equality in (1.3) iff we have equality in Cauchy-Schwarz and $x \cdot y = |x \cdot y|$, which happens iff x is parallel to y with $\lambda \geq 0$. □

The Cross Product

For this section, we restrict our attention to the case $n = 3$.

Before stating the definition of the cross product of two vectors in $\mathbb{R}^3$, we recall that the determinant of a 2×2 matrix is

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc,$$

and for a 3×3 matrix

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}.$$

Definition 2. *For any two vectors $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in \mathbb{R}^3$, we define thie cross product to be*

$$x \times y := \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} \hat{i} - \begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix} \hat{j} + \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \hat{k} \in \mathbb{R}^3, \tag{1.4}$$

where $\hat{i} = (1, 0, 0)$, $\hat{j} = (0, 1, 0)$ and $\hat{k} = (0, 0, 1)$.

Example 6. If $x = (2, 3, 4)$ and $y = (1, 2, 3)$, then

$$x \times y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 3 & 4 \\ 2 & 3 \end{vmatrix} \hat{i} - \begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix} \hat{j} + \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} \hat{k}$$
$$= \hat{i} - 2\hat{j} + \hat{k}$$
$$= (1, -2, 1).$$

Remark.

$$\begin{aligned} \hat{i} \times \hat{i} &= 0, & \hat{i} \times \hat{j} &= \hat{k}, & \hat{i} \times \hat{k} &= -\hat{j}, \\ \hat{j} \times \hat{i} &= -\hat{k}, & \hat{j} \times \hat{j} &= 0, & \hat{j} \times \hat{k} &= \hat{i}, \\ \hat{k} \times \hat{i} &= \hat{j}, & \hat{k} \times \hat{j} &= -\hat{i}, & \hat{k} \times \hat{k} &= 0. \end{aligned}$$

The cross product satisfies the ***right hand rule****.*

Properties of the Cross Product

Let $x, y, z \in \mathbb{R}^3$ and $\alpha, \beta \in \mathbb{R}$.

a) $x \times y = -y \times x$

b) $(\alpha x + \beta y) \times z = \alpha(x \times z) + \beta(y \times z)$.

c) $(x \times y) \cdot x = (x \times y) \cdot y = 0$, so $x \times y$ is orthogonal to the plane spanned by x and y.

d) If θ denotes the angle between x and y, then

$$\|x \times y\| = \|x\| \|y\| \sin\theta,$$

which is equal to the area of the parallelogram generated by x and y.

Proof. Properties a)-c) follow follow directly from the definition and are left as an Exercise. To show d) we begin by expanding the term $(x \times y) \cdot (x \times y)$, and find that

$$\begin{aligned} \|x \times y\|^2 &= \|x\|^2 \|y\|^2 - (x \cdot y)^2 \\ &= \|x\|^2 \|y\|^2 (1 - \cos^2(\theta)) \\ &= \|x\|^2 \|y\|^2 \sin^2(\theta). \end{aligned}$$

Note that the area is given by the base $\|x\|$ times the height $\|y\| \sin\theta$. □

Remark. *In the case that $x, y \in \mathbb{R}^2$, we see that*

$$x \times y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_1 & x_2 & 0 \\ y_1 & y_2 & 0 \end{vmatrix} = \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \hat{k}, \tag{1.5}$$

and so $\|x \times y\| = x_1 y_2 - x_2 y_1$.

Triple Product

For any $x, y, z \in \mathbb{R}^3$, we define their triple product as $(x \times y) \cdot z \in \mathbb{R}$.

Unravelling the definition, we see that

$$(x \times y) \cdot z = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} \cdot \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{vmatrix} z_1 & z_2 & z_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}. \tag{1.6}$$

It follows from (1.6) that

$$(x \times y) \cdot z = (y \times z) \cdot x = (z \times x) \cdot y = -(y \times x) \cdot z = -(z \times y) \cdot x = -(x \times z) \cdot y$$

The triple product has the following geometric interpretation.

Lemma 7. *$|(x \times y) \cdot z|$ is the volume of the parallelepiped spanned by x, y and z.*

Proof. Let α denote the angle formed between the vector $x \times y$ and z.

Note that, after possibly replacing $x \times y$ with $y \times x$, we may assume that $\alpha \in [0, \frac{\pi}{2}]$.

Since $(x \times y) \cdot z = \|x \times y\| \|z\| \cos \alpha$, and our parallelepiped has base area $\|x \times y\|$ and height $\|z\| \cos \alpha$, the result follows. □

Remark. *The triple product $(x \times y) \cdot z = 0$ iff the volume of the parallelepiped vanishes iff $\{x, y, z\}$ are linearly dependent.*

Example 8. Consider the parallelepiped spanned by the vectors $(1, 1, 0), (0, 1, 0), (1, 1, 1) \in \mathbb{R}^3$. The triple product of these three vectors is

$$\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 1,$$

and therefore the parallelepiped has unit volume.

1.2 Affine Subspaces