MATH2010E

Tutorial 9 Applications of Chain Rule

1. Exercise 14.6

Q2 Find equations for the

(a) tangent plane and

(b) normal line at the point P_0 on the given surface

$$\underbrace{x^2+y^2-z^2}_{f(x,y,z)} = 18 , \qquad P_0(3,5,-4)$$

surface $f(x,y,z) = c$

Curve $\vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j} + z(t)\,\vec{k}$

$$\underset{\substack{\uparrow \\ \because f \text{ is constant}}}{0} = \frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \underbrace{\vec{r}'(t)}_{\text{tangent vector}} \qquad \text{(chain rule)}$$

$$\boxed{\begin{array}{l} \nabla f \perp \text{ tangent plane} \\ \nabla f \text{ parallel to normal line.} \end{array}}$$

$f_x(P_0)(x-x_0) + f_y(P_0)(y-y_0) + f_z(P_0)(z-z_0) = 0$

$x = x_0 + f_x(P_0)t$,

$y = y_0 + f_y(P_0)t$,

$z = z_0 + f_z(P_0)t$

(a) $\nabla f = (2x, 2y, -2z)$

$\nabla f(3,5,-4) = (6, 10, 8)$

$\Rightarrow$ Tangent plane :

$$6(x-3) + 10(y-5) + 8(z+4) = 0$$

$$3x + 5y + 4z = 18$$

(b) Normal line :

$$x = 3+6t, \quad y = 5+10t, \quad z = -4+8t$$

2. Implicit Differentiation

Find the values of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at the points

$$xe^y + ye^z + 2\ln x - 2 - 3\ln 2 = 0, \quad (1, \ln 2, \ln 3)$$

z is regarded as a function $z = z(x,y)$ of independent variables x, y locally near $(1, \ln 2, \ln 3)$.

let $F(x,y,z) = xe^y + ye^z + 2\ln x - 2 - 3\ln 2 = 0$

$$\frac{\partial F}{\partial x} = e^y + ye^z \frac{\partial z}{\partial x} + \frac{2}{x} = 0 \qquad \text{(treat } y \text{ as a constant)}$$

$$\frac{\partial F}{\partial y} = xe^y + e^z + ye^z \frac{\partial z}{\partial y} = 0 \qquad \text{(treat } x \text{ as a constant)}$$

$$\Rightarrow \left.\frac{\partial F}{\partial x}\right|_{(1,\ln 2,\ln 3)} = e^{\ln 2} + (\ln 2)e^{\ln 3} \left.\frac{\partial z}{\partial x}\right|_{(1,\ln 2,\ln 3)} + \frac{2}{1} = 0$$

$$\left.\frac{\partial F}{\partial y}\right|_{(1,\ln 2,\ln 3)} = 1\cdot e^{\ln 2} + e^{\ln 3} + (\ln 2)e^{\ln 3} \left.\frac{\partial z}{\partial y}\right|_{(1,\ln 2,\ln 3)} = 0$$

$$\Rightarrow \left.\frac{\partial z}{\partial x}\right|_{(1,\ln 2,\ln 3)} = \frac{-4}{3\ln 2}$$

$$\left.\frac{\partial z}{\partial y}\right|_{(1,\ln 2,\ln 3)} = \frac{-5}{3\ln 2}$$

3. Suppose that the partial derivatives of a function $f(x,y,z)$ at points on the helix $x = \cos t$, $y = \sin t$, $z = t$ are $f_x = \cos t$, $f_y = \sin t$, $f_z = t^2 + t - 2$.

At what points on the curve, if any, can f take on extreme values?

On extreme values : $\frac{df}{dt} = 0$

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \quad \text{(Chain Rule)}$$

$$= (\cos t)(-\sin t) + (\sin t)(\cos t) + (t^2+t-2)(1)$$

$$= t^2 + t - 2$$

$$\Rightarrow t^2 + t - 2 = 0$$

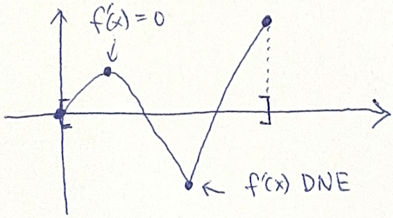
$$t = -2 \text{ or } 1$$

when $t = -2$, $(x, y, z) = (\cos(-2), \sin(-2), -2)$

when $t = 1$, $(x, y, z) = (\cos 1, \sin 1, 1)$

Finding extrema (absolute max / min) :

One variable :



Extrema can only occur at

- critical points : $f'(x) = 0$, $f'(x)$ DNE
- boundary points

comparing values of f at these points for finding extrema.

Multivariable :

Critical points :

- $\nabla f(\vec{x}) = \vec{0}$ (i.e. $\frac{\partial f}{\partial x_i}(\vec{x}) = 0$ for all i)
- $\nabla f(\vec{x})$ DNE (i.e. $\frac{\partial f}{\partial x_i}(\vec{x})$ DNE for some i)