MATH 2010E

Tutorial 8 Chain Rule

Chapter 14 Additional and Advanced Exercises (P. 879)

Q3 Leibniz's Rule:

If f is continuous on $[a,b]$ and if $u(x)$ and $v(x)$ are differentiable functions of x whose values lie in $[a,b]$, then

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)\,dt = f(v(x))\frac{dv}{dx} - f(u(x))\frac{du}{dx}.$$

Prove the rule by setting

$$g(u,v) = \int_u^v f(t)\,dt, \quad u = u(x), \quad v = v(x)$$

and calculating $\frac{dg}{dx}$ with the Chain Rule.

$$\frac{dg}{dx} = \frac{\partial g}{\partial u}\frac{du}{dx} + \frac{\partial g}{\partial v}\frac{dv}{dx} \qquad \text{(Chain Rule)}$$

$$= \left(\frac{\partial}{\partial u}\int_u^v f(t)\,dt\right)\frac{du}{dx} + \left(\frac{\partial}{\partial v}\int_u^v f(t)\,dt\right)\frac{dv}{dx}$$

$$= \left(-\frac{\partial}{\partial u}\int_v^u f(t)\,dt\right)\frac{du}{dx} + \left(\frac{\partial}{\partial v}\int_u^v f(t)\,dt\right)\frac{dv}{dx}$$

$$= -f(u)\frac{du}{dx} + f(v)\frac{dv}{dx}$$

$$= f(v(x))\frac{dv}{dx} - f(u(x))\frac{du}{dx}$$

Q4 Suppose that f is a twice-differentiable function of r, that $r = \sqrt{x^2+y^2+z^2}$, and that

$$f_{xx} + f_{yy} + f_{zz} = 0.$$

Show that for some constants a and b,

$$f(r) = \frac{a}{r} + b.$$

By chain rule,

$$f_x = \frac{df}{dr} \cdot \frac{\partial r}{\partial x}$$

$$f_{xx} = \left(\frac{d^2f}{dr^2}\right)\frac{\partial r}{\partial x} \cdot \frac{\partial r}{\partial x} + \frac{df}{dr}\frac{\partial^2 r}{\partial x^2}$$

$$= \left(\frac{d^2f}{dr^2}\right)\left(\frac{\partial r}{\partial x}\right)^2 + \frac{df}{dr}\frac{\partial^2 r}{\partial x^2}$$

Similarly, $f_{yy} = \left(\frac{d^2f}{dr^2}\right)\left(\frac{\partial r}{\partial y}\right)^2 + \frac{df}{dr}\frac{\partial^2 r}{\partial y^2}$

$$f_{zz} = \left(\frac{d^2f}{dr^2}\right)\left(\frac{\partial r}{\partial z}\right)^2 + \frac{df}{dr}\frac{\partial^2 r}{\partial z^2}$$

$$\frac{\partial r}{\partial x} = \frac{1}{2}\frac{1}{\sqrt{x^2+y^2+z^2}}(2x) = \frac{x}{\sqrt{x^2+y^2+z^2}}$$

$$\frac{\partial^2 r}{\partial x^2} = \frac{1\sqrt{x^2+y^2+z^2} - x\frac{x}{\sqrt{x^2+y^2+z^2}}}{x^2+y^2+z^2} = \frac{y^2+z^2}{(\sqrt{x^2+y^2+z^2})^3}$$

Similarly, $\frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2+z^2}}$, $\frac{\partial^2 r}{\partial y^2} = \frac{x^2+z^2}{(\sqrt{x^2+y^2+z^2})^3}$

$\frac{\partial r}{\partial z} = \frac{z}{\sqrt{x^2+y^2+z^2}}$, $\frac{\partial^2 r}{\partial z^2} = \frac{x^2+y^2}{(\sqrt{x^2+y^2+z^2})^3}$

$$f_{xx} + f_{yy} + f_{zz} = 0$$

$$\Rightarrow \left(\frac{d^2f}{dr^2}\right)\frac{x^2}{x^2+y^2+z^2} + \left(\frac{df}{dr}\right)\frac{y^2+z^2}{(\sqrt{x^2+y^2+z^2})^3}$$

$$+ \left(\frac{d^2f}{dr^2}\right)\frac{y^2}{x^2+y^2+z^2} + \left(\frac{df}{dr}\right)\frac{x^2+z^2}{(\sqrt{x^2+y^2+z^2})^3}$$

$$+ \left(\frac{d^2f}{dr^2}\right)\frac{z^2}{x^2+y^2+z^2} + \left(\frac{df}{dr}\right)\frac{x^2+y^2}{(\sqrt{x^2+y^2+z^2})^3} = 0$$

$$\Rightarrow \left(\frac{d^2f}{dr^2}\right) + \left(\frac{df}{dr}\right)\left(\frac{2}{\sqrt{x^2+y^2+z^2}}\right) = 0$$

$$\frac{d^2f}{dr^2} + \frac{2}{r}\frac{df}{dr} = 0$$

$$\frac{df'}{dr} = -\frac{2}{r}f' \quad , \quad \text{where } f' = \frac{df}{dr}$$

$$\int \frac{1}{f'}\,df' = \int -\frac{2}{r}\,dr$$

$$\ln f' = -2\ln r + C'$$

$$f' = e^{-2\ln r + C'} = Cr^{-2} \quad \text{for some constant } C, \ C = e^{C'}$$

$$\Rightarrow f = \int C r^{-2}\,dr$$

$$= C\frac{r^{-1}}{-1} + b \quad \text{for some constant } b$$

$$= \frac{a}{r} + b \quad \text{for some constants } a, b, \ a = -C$$

Q5 A function $f(x,y)$ is homogeneous of degree n (n a nonnegative integer) if $f(tx,ty) = t^n f(x,y)$ for all t, x, and y. For such a function (sufficiently differentiable), prove that

a. $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x,y)$

b. $x^2\left(\frac{\partial^2 f}{\partial x^2}\right) + 2xy\left(\frac{\partial^2 f}{\partial x\partial y}\right) + y^2\left(\frac{\partial^2 f}{\partial y^2}\right) = n(n-1)f$

9. Let $u = tx$, $v = ty$,

$$w = f(u,v) = f(tx, ty) = t^n f(x,y).$$

$$\frac{\partial w}{\partial t} = n t^{n-1} f(x,y)$$

$$\parallel$$

$$\frac{\partial w}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial t} = \frac{\partial w}{\partial u}\cdot x + \frac{\partial w}{\partial v}\cdot y$$

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial w}{\partial u}\cdot t + \frac{\partial w}{\partial v}\cdot 0$$

$$\Rightarrow \frac{\partial w}{\partial u} = \frac{1}{t}\frac{\partial w}{\partial x}$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial w}{\partial u}\cdot 0 + \frac{\partial w}{\partial v}\cdot t$$

$$\Rightarrow \frac{\partial w}{\partial v} = \frac{1}{t}\frac{\partial w}{\partial y}$$

$$\Rightarrow n t^{n-1} f(x,y) = \frac{1}{t}\frac{\partial w}{\partial x}\cdot x + \frac{1}{t}\frac{\partial w}{\partial y}\cdot y$$

$$= \frac{x}{t}\frac{\partial w}{\partial x} + \frac{y}{t}\frac{\partial w}{\partial y}$$

When $t = 1$, $u = x$, $v = y$, $w = f(x,y)$,

$$\Rightarrow n f(x,y) = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}$$

b. From (a), $n t^{n-1} f(x,y) = x \frac{\partial w}{\partial u} + y \frac{\partial w}{\partial v}$

Differentiate with respect to t again,

$$n(n-1) t^{n-2} f(x,y) = x\left(\frac{\partial^2 w}{\partial u^2} \frac{\partial u}{\partial t} + \frac{\partial^2 w}{\partial v \partial u} \frac{\partial v}{\partial t} \right) + y\left(\frac{\partial^2 w}{\partial u \partial v} \frac{\partial u}{\partial t} + \frac{\partial^2 w}{\partial v^2} \frac{\partial v}{\partial t} \right)$$

$$= x^2 \frac{\partial^2 w}{\partial u^2} + 2xy \frac{\partial^2 w}{\partial u \partial v} + y^2 \frac{\partial^2 w}{\partial v^2}$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right) \overset{(a)}{=} \frac{\partial}{\partial x}\left(t \frac{\partial w}{\partial u}\right) = t\left(\frac{\partial^2 w}{\partial u^2} \underbrace{\frac{\partial u}{\partial x}}_{t} + \frac{\partial^2 w}{\partial v \partial u} \underbrace{\frac{\partial v}{\partial x}}_{0} \right)$$

$$= t^2 \frac{\partial^2 w}{\partial u^2}$$

Similarly, $\frac{\partial^2 w}{\partial y^2} = t^2 \frac{\partial^2 w}{\partial v^2}$.

$$\frac{\partial^2 w}{\partial y \partial x} = t^2 \frac{\partial^2 w}{\partial v \partial u}.$$

$$\Rightarrow n(n-1) t^{n-2} f(x,y) = \frac{x^2}{t^2} \frac{\partial^2 w}{\partial x^2} + \frac{2xy}{t^2} \frac{\partial^2 w}{\partial y \partial x} + \frac{y^2}{t^2} \frac{\partial^2 w}{\partial y^2}, \quad t \neq 0$$

When $t = 1$, $w = f(x,y)$,

$$\Rightarrow n(n-1) f(x,y) = x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2}$$