MATH 2010E TUT07

1. (*12 pts*) Answer the following questions.

   (a) Find the equation of plane Π passing through the point $(2, 3, -1)$ and parallel to the plane $3x - 4y + 7z = 1$.

   (b) Find the distance between the two planes in part (a).

   (c) Find the angle between the plane Π and the plane $8x + 3y - z = 2$.

Ans: a) Since the plane is parallel to $3x - 4y + 7z = 1$,
it has equ $3x - 4y + 7z = d$ for some $d \in \mathbb{R}$.
Put $(x, y, z) = (2, 3, -1)$, we have

$$3(2) - 4(3) + 7(-1) = d \implies d = -13.$$

Therefore, equ of Π is $3x - 4y + 7z = -13$.

b) Method I: By distance formula,

$$\text{distance} = \left| \frac{1 - (-13)}{\sqrt{3^2 + (-4)^2 + 7^2}} \right| = \frac{14}{\sqrt{74}} = \frac{7\sqrt{74}}{37}$$

Method II: Let $A = (2, 3, -1)$ on Π

$B = (-1, -1, 0)$ on $\Pi' : 3x - 4y + 7z = 1$

$\vec{n} = (3, -4, 7)$ common normal.

$$\text{Distance between } \Pi, \Pi' = \|\mathrm{Proj}_{\vec{n}} \overrightarrow{BA}\| = \frac{|(3,4,-1) \cdot (3,-4,7)|}{\|(3,-4,7)\|}$$

$$= \frac{|9 - 16 - 7|}{\sqrt{3^2 + 4^2 + 7^2}} = \frac{14}{\sqrt{74}}$$

c) Angle between planes = angle between normals

$$= \cos^{-1}\left(\frac{(8,3,-1) \cdot (3,-4,7)}{\|(8,3,-1)\| \, \|(3,-4,7)\|} \right)$$

$$= \cos^{-1}\left(\frac{5}{74}\right) \in \left[0, \frac{\pi}{2}\right].$$

$$\approx 86.13^\circ$$

2. (*6 pts*) Compute the arclength of the curve $\gamma(t) = (t^2, 2t, \ln t)$ for $1 \le t \le 5$.

Ans : $r'(t) = \left(2t, 2, \frac{1}{t}\right) \quad , 1 \le t \le 5 .$

$$\begin{aligned}
\text{Arclength} &= \int_1^5 \|\gamma'(t)\| \, dt \\
&= \int_1^5 \sqrt{(2t)^2 + 2^2 + \left(\frac{1}{t}\right)^2} \, dt \\
&= \int_1^5 \sqrt{\left(2t + \frac{1}{t}\right)^2} \, dt \\
&= \int_1^5 \left(2t + \frac{1}{t}\right) dt \quad > 0 \\
&= \left[t^2 + \ln t\right]_1^5 \\
&= 25 + \ln 5 - 1 - \ln 1 \\
&= 24 + \ln 5
\end{aligned}$$

3. (*10 pts*) Evaluate the following limits or show they do not exist.

(a) $\lim_{(x,y)\to(0,0)} \frac{x^2y - y^3}{x^2+y^2}$

Ans: a) $\forall (x,y) \neq (0,0)$,

$$0 \leq \left| \frac{x^2y - y^3}{x^2+y^2} \right| \leq \frac{x^2}{x^2+y^2}|y| + \frac{y^2}{x^2+y^2}|y|$$

$$\leq 2|y|.$$

Since $\lim_{(x,y)\to(0,0)} 2|y| = 0$, it follows from Squeeze Thm that

$$\lim_{(x,y)\to(0,0)} \frac{x^3y - y^3}{x^2+y^2} = 0.$$

(b) $\lim_{(x,y)\to(0,0)} \frac{x^3y - x^2y - 3xy^2}{x^4+y^2}$

b) Consider limits along different paths.

$$\lim_{\substack{(x,y)\to(0,0)\\ x=0}} \frac{x^3y - x^2y - 3xy^2}{x^4+y^2} = \lim_{y\to 0} \frac{0}{y^2} = 0.$$

while

$$\lim_{\substack{(x,y)\to(0,0)\\ y=x^2}} \frac{x^3y - x^2y - 3xy^2}{x^4+y^2} = \lim_{x\to 0} \frac{x^5 - x^4 - 3x^5}{x^4+x^4}$$

$$= \lim_{x\to 0} \frac{1}{2}(-2x-1) = -\frac{1}{2}.$$

Since the limits are different, $\lim_{(x,y)\to(0,0)} \frac{x^3y - x^2y - 3xy^2}{x^4+y^2}$ DNE

4. (*10 pts*) Let $f(x,y) = \dfrac{e^{xy+6}}{1+4x+3y}$.

(a) Find df, the differential of f.

(b) Use the result of (a) to approximate the change in f when (x,y) changes from $(-2,3)$ to $(-1.9, 2.95)$.

Aw: a) $\dfrac{\partial f}{\partial x} = \dfrac{1}{(1+4x+3y)^2}\left[(1+4x+3y)(ye^{xy+6}) - e^{xy+6}(4)\right]$

$= \dfrac{(1+4x+3y)y - 4}{(1+4x+3y)^2} e^{xy+6}$

$\dfrac{\partial f}{\partial y} = \dfrac{1}{(1+4x+3y)^2}\left[(1+4x+3y)(xe^{xy+6} - e^{xy+6}(3)\right]$

$= \dfrac{(1+4x+3y)x - 3}{(1+4x+3y)^2} e^{xy+6}$

Hence, $df = \dfrac{\partial f}{\partial x} dx + \dfrac{\partial f}{\partial y} dy$

$= \dfrac{[(1+4x+3y)y - 4]e^{xy+6}}{(1+4x+3y)^2} dx + \dfrac{[(1+4x+3y)x - 3]e^{xy+6}}{(1+4x+3y)^2} dy$

b) Put $(x,y) = (-2,3)$. Then

$dx = -1.9 - (-2) = 0.1$, $dy = 2.95 - 3 = -0.05$

$\dfrac{\partial f}{\partial x} = \dfrac{[(2)(3) - 4]e^0}{2^2} = \dfrac{1}{2}$

$\dfrac{\partial f}{\partial y} = \dfrac{[(2)(-2) - 3]e^0}{2^2} = -\dfrac{7}{4}$.

Hence $\Delta f \approx df = \frac{1}{2}(0.1) - \frac{7}{4}(-0.05) = 0.05 + 0.0875$

$= 0.1375$

5. (*10 pts*) Lef $f(x,y) = \ln(30 - 10x + x^2 + y^2)$.

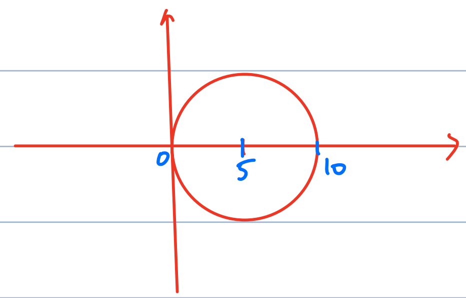
(a) Draw the level set of f through the point $(2,4)$. Label all its intercept(s).

(b) Find the direction where f decreases most rapidly at the point $(2,4)$.

Ans: a) $f(2,4) = \ln(30 - 20 + 4 + 16) = \ln 30$

Hence

$$f(x,y) = f(2,4)$$
$$\ln(30 - 10x + x^2 + y^2) = \ln 30$$
$$30 - 10x + x^2 + y^2 = 30$$
$$x^2 + y^2 - 10x = 0$$
$$(x-5)^2 + y^2 = 5^2$$



The level set is a circle centered at $(5,0)$ with radius 5

b) $$\frac{\partial f}{\partial x} = \frac{-10 + 2x}{30 - 10x + x^2 + y^2}$$

$$\frac{\partial f}{\partial y} = \frac{2y}{30 - 10x + x^2 + y^2}$$

$$\Rightarrow \nabla f = \left(\frac{-10 + 2x}{30 - 10x + x^2 + y^2} , \frac{2y}{30 - 10x + x^2 + y^2} \right)$$

So

$$-\nabla f(2,4) = -\left(\frac{-6}{30} , \frac{8}{30} \right)$$
$$= \left(\frac{1}{5} , -\frac{4}{15} \right)$$

The direction where f decreases most rapidly at $(2,4)$:

$$\frac{-\nabla f(2,4)}{\|\nabla f(2,4)\|} = \left(\frac{3}{5} , -\frac{4}{5} \right)$$

7. (*22 pts*) Let

$$f(x,y) = \begin{cases} \sqrt[3]{xy^2}\sin\dfrac{x}{y} & \text{if } \quad y \neq 0; \\ \\ 0 & \text{if } \quad y = 0. \end{cases}$$

(a) Show that f is continuous at $(0,0)$.

(b) Show that $\dfrac{\partial f}{\partial x}(0,0) = 0$ and $\dfrac{\partial f}{\partial y}(0,0) = 0$.

(c) Let $\mathbf{u} = \left(-\dfrac{3}{5}, \dfrac{4}{5}\right)$. Compute the directional derivative $\nabla_{\mathbf{u}} f(0,0) = D_{\mathbf{u}} f(0,0)$.

(d) Determine all the point(s) for which f is differentiable? Prove your assertion.

Ans: a) Note $\forall\, (x,y) \neq (0,0)$,

$$0 \leq |f(x,y)| = \begin{cases} |\sqrt[3]{xy^2}|\,|\sin\frac{x}{y}| & , \text{ if } y \neq 0 \\ 0 & , \text{ if } y = 0 \end{cases}$$

$$\leq |\sqrt[3]{xy^2}|$$

Since $\lim_{(x,y)\to(0,0)} |\sqrt[3]{xy^2}| = 0$, it follows from Sandwich Thm that

$\lim_{(x,y)\to(0,0)} |f(x,y)| = 0$.

Thus $\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$

Hence f is cts at $(0,0)$

Caution: $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \sqrt[3]{xy^2}\sin\frac{x}{y}$ X

b) $$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0.$$

and $$\frac{\partial f}{\partial y}(0,0) = \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = \lim_{k\to 0} \frac{\sqrt[3]{0\cdot k^2}-0}{k} = 0.$$

c) $$D_{\vec{u}} f(0,0) = \lim_{t\to 0} \frac{f(t\vec{u}) - f(\vec{0})}{t}$$

$t\vec{u} = \left(-\frac{3}{5}t, \frac{4}{5}t\right)$ $\neq 0$ if $t \neq 0$

$$= \lim_{t\to 0} \frac{1}{t}\left[\sqrt[3]{\left(-\tfrac{3}{5}t\right)\left(\tfrac{4}{5}t\right)^2}\sin\left(\frac{-3t/5}{4t/5}\right) - 0\right]$$

$$= \lim_{t\to 0} \sqrt[3]{\frac{-48}{125}}\cdot\cancel{\frac{t}{t}}\sin\left(-\frac{3}{4}\right) = -\frac{2\sqrt[3]{6}}{5}\sin\left(-\frac{3}{4}\right)$$

Caution: $D_{\vec{u}} f(0,0) \overset{?}{=} \nabla f(0,0)\cdot\vec{u}$ X

need f diff. at $(0,0)$

7. (*22 pts*) Let

$$f(x,y)=\begin{cases}\sqrt[3]{xy^2}\sin\dfrac{x}{y} & \text{if}\quad y\neq 0;\\[2ex] 0 & \text{if}\quad y=0.\end{cases}$$

(a) Show that f is continuous at $(0,0)$.

(b) Show that $\dfrac{\partial f}{\partial x}(0,0)=0$ and $\dfrac{\partial f}{\partial y}(0,0)=0$.

(c) Let $\mathbf{u}=\left(-\dfrac{3}{5},\dfrac{4}{5}\right)$. Compute the directional derivative $\nabla_{\mathbf{u}}f(0,0)=D_{\mathbf{u}}f(0,0)$.

(d) Determine all the point(s) for which f is differentiable? Prove your assertion.

Ans: d). 1) Since $\nabla f(0,0)\cdot\vec{u}=\vec{0}\cdot\vec{u}=0\neq D_{\vec{u}}f(0,0)$,

f is not diff. at $(0,0)$.

2) For $(x,0)$, $x\neq 0$.

$$\frac{\partial f}{\partial y}(x,0)=\lim_{k\to 0}\frac{f(x,k)-f(x,0)}{k}$$

$$=\lim_{k\to 0}\frac{1}{k}\left[\sqrt[3]{xk^2}\sin\frac{x}{k}-0\right]$$

$$=\lim_{k\to 0}\sqrt[3]{\frac{x}{k}}\sin\frac{x}{k}$$

which DNE (Take $k_n=\dfrac{x}{2n\pi}$, $k_n'=\dfrac{x}{2n\pi+\frac{\pi}{2}}$

Then $\lim\limits_{n\to\infty}k_n=\lim\limits_{n\to\infty}k_n'=0$

but $\lim\limits_{n\to\infty}\sqrt[3]{\dfrac{x}{k_n}}\sin\dfrac{x}{k_n}=0$, $\lim\limits_{n\to\infty}\sqrt[3]{\dfrac{x}{k_n'}}\sin\dfrac{x}{k_n'}=+\infty$)

So f is not diff. at $(x,0)$, $x\neq 0$.

3) For $(0,y)$, $y\neq 0$,

$f(0,y)=0 \Rightarrow \dfrac{\partial f}{\partial y}(0,y)=0$.

Also, $$\frac{\partial f}{\partial x}(0,y)=\lim_{h\to 0}\frac{1}{h}\left[f(h,y)-f(0,y)\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\sqrt[3]{hy^2}\sin\frac{h}{y}\right]$$

$$=\lim_{h\to 0}\sqrt[3]{\frac{y^2}{h^2}}\sin\left(\frac{h}{y}\right)$$

$$=\lim_{h\to 0}\sqrt[3]{\frac{h}{y}}\cdot\frac{\sin(h/y)}{h/y}$$

$$=0\cdot 1=0.$$

For $x \neq 0,\ y \neq 0$,

$$f(x,y) = \sqrt[3]{xy^2}\,\sin\left(\frac{x}{y}\right)$$

$$\Rightarrow \quad \frac{\partial f}{\partial x}(x,y) = \frac{1}{3}x^{-\frac{2}{3}}y^{\frac{2}{3}}\sin\frac{x}{y} + \sqrt[3]{xy^2}\cdot\frac{1}{y}\cos\left(\frac{x}{y}\right)$$

$$= \underbrace{\frac{1}{3}\,\frac{\sin(x/y)}{x/y}\cdot x^{\frac{1}{3}}y^{-\frac{1}{3}}}_{\to 0} + \underbrace{x^{\frac{1}{3}}y^{-\frac{1}{3}}\cos\left(\frac{x}{y}\right)}_{\to 0} \quad \text{as } x \to 0$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{2}{3}x^{\frac{1}{3}}y^{-\frac{1}{3}}\sin\left(\frac{x}{y}\right) + \sqrt[3]{xy^2}\left(-xy^{-2}\right)\cos\left(\frac{x}{y}\right)$$

$$= \underbrace{\frac{2}{3}x^{\frac{1}{3}}y^{-\frac{1}{3}}\sin\left(\frac{x}{y}\right)}_{\to 0} - \underbrace{x^{\frac{4}{3}}y^{-\frac{4}{3}}\cos\left(\frac{x}{y}\right)}_{\to 0} \quad \text{as } x \to 0$$

Now both $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ are cts on $\{(x,y) : x \neq 0, y \neq 0\}$.

and for $y_0 \neq 0$,

$$\left.\begin{aligned} \lim_{(x,y)\to(0,y_0)} \frac{\partial f}{\partial x}(x,y) &= 0 = \frac{\partial f}{\partial x}(0,y_0) \\ \lim_{(x,y)\to(0,y_0)} \frac{\partial f}{\partial y}(x,y) &= 0 = \frac{\partial f}{\partial y}(0,y_0) \end{aligned}\right\} \Rightarrow \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \text{ cts at } (0,y_0)$$

open set

Hence f is C^1 on $\{(x,y) : y \neq 0\}$.

$\Rightarrow$ f is diff. on $\{(x,y) : y \neq 0\}$.

Combining 1), 2), 3), the set where f is diff. is $\{(x,y) : y \neq 0\}$.

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