MATH2010E TUT06

Thm Let $\Omega \subseteq \mathbb{R}^n$ be open.

If f is C^1 on Ω,

Then f is differentiable on Ω.

$\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}$ all are cts on Ω

$\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}$ cts on $\Omega \underset{\not\Leftarrow}{\Rightarrow} f$ is diff on $\Omega \underset{\not\Leftarrow}{\Rightarrow} \begin{cases} \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \text{ exist on } \Omega \\ f \text{ cts on } \Omega \end{cases}$

$f(x,y) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$ $\qquad f(x,y) = \sqrt{|xy|}$

Pf: MVT + continuity of $\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}$ //

Def (Directional Derivative)

Let $\Omega \subseteq \mathbb{R}^n$ be open, $\vec{a} \in \Omega$, $f: \Omega \to \mathbb{R}$.
Let $\vec{u} \in \mathbb{R}^n$ be a unit vector.
The directional derivative of f in the direction of $\vec{u}$ at $\vec{a}$ is defined by

$$D_{\vec{u}} f(\vec{a}) = \lim_{t \to 0} \frac{f(\vec{a} + t\vec{u}) - f(\vec{a})}{t}$$

= Rate of change of f in the direction of $\vec{u}$ at the pt. $\vec{a}$.

Thm Suppose f is differentiable at $\vec{a}$.
Let $\vec{u} \in \mathbb{R}$ be a unit vector.
Then $D_{\vec{u}} f(\vec{a})$ exists and

$$D_{\vec{u}} f(\vec{a}) = \nabla f(\vec{a}) \cdot \vec{u}$$

Remark: 1)

f diff. $\Rightarrow$ All directional derivatives exist $\Rightarrow$ All partial derivatives exist.

$\not\Leftarrow$ $\qquad\qquad\qquad\qquad\qquad$ $\not\Leftarrow$

2) By Cauchy-Schwarz Inequality,

$= \text{ if } \vec{u} = \nabla f / \|\nabla f\|$

$$-\|\nabla f\| \leq D_{\vec{u}} f \leq \|\nabla f\| \cdot \|\vec{u}\| = \|\nabla f\|$$

$\Rightarrow$ f increases most rapidly in the direction ∇f
 decreases $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $-\nabla f$

Ex 1 Find the directional derivative of each of the following functions at the given point and direction:

(a) $x^2 + y^3 + z^4$, $(3, 2, 1)$; $(-1, 0, 4)/\sqrt{17}$.
(b) $e^{xy} + \sin(x^2 + y^2)$, $(1, -3)$; $(1, 1)/\sqrt{2}$.

Ans: b)

$$g(x,y) = e^{xy} + \sin(x^2+y^2)$$
$$g_x = ye^{xy} + 2x\cos(x^2+y^2)$$
$$g_y = xe^{xy} + 2y\cos(x^2+y^2)$$

Since g, g_x, g_y are cts on $\mathbb{R}^2$, g is C^1 on $\mathbb{R}^2$ and so g is diff. on $\mathbb{R}^2$.

Let $\vec{v} = (1,1)/\sqrt{2}$. Then, by Thm,

$$D_{\vec{v}}\, g(1,-3) = \nabla g(1,-3)\cdot\vec{v}$$
$$= (-3e^{-3} + 2\cos 10,\ e^{-3} - 6\cos 10)\cdot\frac{(1,1)}{\sqrt{2}}$$
$$= \frac{1}{\sqrt{2}}(-2e^{-3} - 4\cos 10)$$
$$= -\sqrt{2}(e^{-3} + 2\cos 10)$$

Ex 2

Let $\mathbf{u} = (1/\sqrt{2}, 1/\sqrt{2})$, $\mathbf{v} = (1/\sqrt{2}, -1/\sqrt{2})$ and $f : \mathbb{R}^2 \to \mathbb{R}$ be differentiable at $(2,3)$. Suppose $D_{\mathbf{u}}f(2,3) = 3$ and $D_{\mathbf{v}}f(2,3) = -4$. Find the direction where f decreases most rapidly at $(2,3)$.

Ans: Note f decreases most rapidly at $(2,3)$ in the direction $-\nabla f(2,3)/\|\nabla f(2,3)\|$

Note
$$D_{\vec{u}}f(2,3) = \nabla f(2,3)\cdot\vec{u}$$
$$D_{\vec{v}}f(2,3) = \nabla f(2,3)\cdot\vec{v}$$

Let $\nabla f(2,3) = (a,b)$. Then
$$\begin{cases} \frac{1}{\sqrt{2}}a + \frac{1}{\sqrt{2}}b = 3 \\ \frac{1}{\sqrt{2}}a - \frac{1}{\sqrt{2}}b = -4. \end{cases}$$
$$\Longleftrightarrow \quad A\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$$
where
$$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$
Since $A\cdot A = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$,

we have
$$\begin{pmatrix} a \\ b \end{pmatrix} = A\begin{pmatrix} 3 \\ -4 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{7}{\sqrt{2}} \end{pmatrix}$$

So $\nabla f(2,3) = \left(-\frac{1}{\sqrt{2}}, \frac{7}{\sqrt{2}}\right)$

The required direction is
$$-\nabla f(2,3)/\|\nabla f(2,3)\|$$
$$= -\left(-\frac{1}{\sqrt{2}}, \frac{7}{\sqrt{2}}\right)\Big/\sqrt{\frac{1}{2}+\frac{49}{2}}$$
$$= \left(\frac{1}{5\sqrt{2}}, -\frac{7}{5\sqrt{2}}\right)$$

Ex 3

.. For the function $f(x_1,x_2) = \begin{cases} \dfrac{|x_1|x_2}{\sqrt{x_1^2+x_2^2}} & \text{if } (x_1,x_2) \neq (0,0) \\ 0 & \text{if } (x_1,x_2) = (0,0), \end{cases}$

(i) Find the directional derivatives at $(0,0)$ (and show they all exist),

(ii) Show that the formula $D_{\underline{v}} f(0,0) = \sum_{i=1}^{2} v_j D_j f(0,0)$ fails for some vectors $\underline{v} \in \mathbb{R}^2$.

(iii) Hence, using (ii) together with the appropriate theorem, prove that f is not differentiable at $(0,0)$.

Ans: (i) Let $\vec{u} = (\cos\theta, \sin\theta)$ be a unit vector.

Then $D_{\vec{u}} f(0,0) = \lim_{t\to 0} \dfrac{f(\vec{0}+t\vec{u}) - f(\vec{0})}{t}$

$$= \lim_{t\to 0} \frac{|t\cos\theta|(t\sin\theta)}{|t|\,t}$$

$$= \lim_{t\to 0} |\cos\theta| \sin\theta$$

$$= |\cos\theta| \sin\theta$$

(ii) $\dfrac{\partial f}{\partial x} = D_{\vec{u}(0)} f(0,0) = 0$

$\dfrac{\partial f}{\partial y} = D_{\vec{u}(\frac{\pi}{2})} f(0,0) = 0$

Take $\vec{v} = \left(\cos\frac{\pi}{4}, \sin\frac{\pi}{4}\right) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

Then $D_{\vec{v}} f(0,0) = \left|\frac{1}{\sqrt{2}}\right| \left(\frac{1}{\sqrt{2}}\right) = \frac{1}{2}$

while $\vec{\nabla} f(0,0) \cdot \vec{v} = (0,0) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = 0 \neq D_{\vec{v}} f(0,0)$

(iii) Since $\vec{\nabla} f(0,0) \cdot \vec{v} \neq D_{\vec{v}} f(0,0)$,

f is NOT diff. at $(0,0)$

Ex 4 Let $\Omega \subseteq \mathbb{R}^2$ be open.
Let $f : \Omega \to \mathbb{R}$ be C^2.
Let $\vec{u}, \vec{v}$ be two unit vectors in $\mathbb{R}^2$.
Show that $D_{\vec{v}}(D_{\vec{u}} f) = D_{\vec{u}}(D_{\vec{v}} f)$ on Ω

Recall: (Clairaut's Thm)
If f_{xy}, f_{yx} exist and are cts on Ω,
then $f_{xy} = f_{yx}$ on Ω

Pf: Note f C^2 on Ω $\Rightarrow$ f C^1 on Ω
$\Rightarrow$ f diff. on Ω

So $D_{\vec{u}} f = \nabla f \cdot \vec{u} = u_1 f_x + u_2 f_y$
Then $\nabla(D_{\vec{u}} f) = (u_1 f_{xx} + u_2 f_{yx}, \; u_1 f_{xy} + u_2 f_{yy})$
cts $\Rightarrow$ $D_{\vec{u}} f$ diff.
Hence $D_{\vec{v}}(D_{\vec{u}} f) = \nabla(D_{\vec{u}} f) \cdot \vec{v}$
$= u_1 v_1 f_{xx} + \underline{u_2 v_1 f_{yx}} + \underline{u_1 v_2 f_{xy}} + u_2 v_2 f_{yy}$
$\parallel \quad \parallel$ by Clairaut's Thm.
Similarly $D_{\vec{u}}(D_{\vec{v}} f) = v_1 u_1 f_{xx} + \underline{v_2 u_1 f_{yx}} + \underline{v_1 u_2 f_{xy}} + v_2 u_2 f_{yy}$
$= D_{\vec{v}}(D_{\vec{u}} f)$

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Ex 5 Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a real-valued function on $\mathbb{R}^2$ such that, for some $\delta > 0$, there exist positive constants $M, C \in \mathbb{R}$ such that:

$$|f(x,y) - M| \leq C \sin^2(\|(x,y)\|)$$

for all $(x,y) \in B_\delta(0,0)$. Is f necessarily differentiable at $(0,0)$? If so, prove it. If not, provide a counter-example.

Ans: $$|f(0,0) - M| \leq C^2 \sin^2(\|(0,0)\|) = 0$$

$$\Rightarrow \quad f(0,0) = M$$

For $0 < |h| < \delta$, we have $(h,0) \in B_\delta(0,0)$ and so

$$\left| \frac{f(h,0) - f(0,0)}{h} \right| \leq \frac{C \sin^2(\|(h,0)\|)}{|h|}$$

$$= C \frac{\sin^2(|h|)}{|h|}$$

$$= C \frac{\sin(|h|)}{|h|} \cdot \sin(|h|)$$

$\nearrow 1$ $\quad$ $\to 0$

Since $\lim_{h \to 0} C \frac{\sin^2(|h|)}{|h|} = 0$, by squeeze thm,

$$\lim_{h \to 0} \left| \frac{f(h,0) - f(0,0)}{h} \right| = 0$$

$$\Rightarrow \quad f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = 0$$

Similarly $f_y(0,0) = 0$.

$$\text{Let } \varepsilon(x,y) = f(x,y) - \underbrace{\cancel{f(0,0)}}_{M} - \underbrace{\cancel{\nabla f(0,0) \cdot [(x,y) - (0,0)]}}_{0}$$

$$= f(x,y) - M$$

Now, $\forall (x,y) \in B_\delta(0,0)$,

$$\left| \frac{\varepsilon(x,y)}{\|(x,y)-(0,0)\|} \right| = \frac{|f(x,y) - M|}{\|(x,y)\|}$$
$$\leq \frac{C \sin^2 \|(x,y)\|}{\|(x,y)\|}$$

Since $\lim_{(x,y)\to(0,0)} C \frac{\sin \|(x,y)\|}{\|(x,y)\|} \cdot \sin \|(x,y)\| = 0$,

it follows from squeeze thm that

$$\lim_{(x,y)\to(0,0)} \frac{\varepsilon(x,y)}{\|(x,y)-(0,0)\|} = 0$$

Hence f is diff. at $(0,0)$ //