# MATH 2010E TUTO 5

**Def.** (Partial Derivatives)

Let $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$, $\Omega$ open.

Let $\vec{x} \in \Omega$.

For $1 \le i \le n$, the i-th partial derivative of f at $\vec{x}$ is defined as

$$f_{x_i}(\vec{x}) = \frac{\partial f}{\partial x_i}(\vec{x}) := \lim_{h \to 0} \frac{f(x_1, \dots, x_i + h, \dots, x_n) - f(x_1, \dots, x_i, \dots, x_n)}{h}$$

$$= \lim_{h \to 0} \frac{f(\vec{x} + h\hat{e}_i) - f(\vec{x})}{h} \qquad \hat{e}_i = (0, \dots, 0, \underset{\substack{\uparrow \\ i\text{-th}}}{1}, 0, \dots, 0)$$

**Remark** $f(x, y)$.

$$f_x = \frac{\partial f}{\partial x} \qquad f_y = \frac{\partial f}{\partial y}$$

$$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} \leftarrow x \text{ first}$$

$$f_{yx} = (f_y)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} \leftarrow y \text{ first.}$$

Ex 1 Let $f(x,y) = \begin{cases} \frac{\sin(xy)}{xy} & \text{if } xy \neq 0 \\ 1 & \text{if } xy = 0 \end{cases}$

Find a) $f_x$ b) $f_{xy}$

Ans: a) At $(x,y)$ where $xy \neq 0$,

$$f_x = \frac{\partial}{\partial x}\left(\frac{\sin(xy)}{xy}\right)$$ $\leftarrow$ treat $y$ as a constant

$$= \frac{1}{y}\,\frac{y\cos(xy)\cdot x - \sin(xy)\cdot(1)}{x^2}$$

$$= \frac{xy\cos(xy) - \sin(xy)}{x^2y}$$

$xy = 0 \iff (x, 0)$ or $(0,y)$, $y \neq 0$

$$f_x(x,0) = \lim_{h\to 0}\frac{f(x+h,0) - f(x,0)}{h}$$

$$= \lim_{h\to 0}\frac{1-1}{h} = 0$$

For $y \neq 0$, $$f_x(0,y) = \lim_{h\to 0} \frac{f(0+h,y)-f(0,y)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{\sin(hy)}{hy} - 1}{h}$$

$$= \lim_{h\to 0} \frac{\sin(hy) - hy}{h^2 y} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{h\to 0} \frac{\frac{\partial}{\partial h}\left(\sin(hy) - hy\right)}{\frac{\partial}{\partial h}\left(h^2 y\right)} \quad \text{(by L'Hopital's rule)}$$

$$= \lim_{h\to 0} \frac{y\cos hy - y}{2hy} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{h\to 0} \frac{\frac{\partial}{\partial h}\left(y\cos hy - y\right)}{\frac{\partial}{\partial h}\left(2hy\right)} \quad \text{(by L'Hopital's rule)}$$

$$= \lim_{h\to 0} \frac{-y^2\sin hy - 0}{2y}$$

$$= 0$$

Hence $$f_x(x,y) = \begin{cases} \dfrac{xy\cos(xy) - \sin(xy)}{x^2 y} & \text{if } xy \neq 0 \\ 0 & \text{if } xy = 0 \end{cases}$$

b) At $(x,y)$ where $xy \neq 0$,

$$f_{xy} = \frac{\partial}{\partial y}\left( \frac{xy\cos(xy) - \sin(xy)}{x^2 y} \right)$$

$$= \frac{\left(-x^2y\sin(xy) + x\cos(xy) - x\cos(xy)\right)\cdot y - \left(xy\cos(xy) - \sin(xy)\right)\cdot 1}{x^2y^2}$$

$$= \frac{-x^2y^2\sin(xy) - xy\cos(xy) + \sin(xy)}{x^2y^2}$$

$$xy = 0 \iff (0,y) \quad \text{or} \quad (x,0), \; x \neq 0$$

$$f_{xy}(0,y) = \lim_{k\to 0} \frac{f_x(0,y+k) - f_x(0,y)}{k} = \lim_{k\to 0} \frac{0-0}{k} = 0$$

For $x \neq 0$, $f_{xy}(x,0) = \lim_{k\to 0} \frac{f_x(x,k) - f_x(x,0)}{k}$

$$= \lim_{k\to 0} \frac{\frac{xk\cos(xk) - \sin(xk)}{x^2k} - 0}{k}$$

$$= \lim_{k\to 0} \frac{xk\cos(xk) - \sin(xk)}{x^2k^2} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{k\to 0} \frac{\cancel{x\cos(xk)} - x^2k\sin(xk) - \cancel{x\cos(xk)}}{2x^2k} \qquad \text{(by L'Hopital's rule)}$$

$$= \lim_{k\to 0} -\frac{1}{2}\sin(xk) = 0$$

Hence

$$f_{xy}(x,y) = \begin{cases} \frac{-x^2y^2\sin(xy) - xy\cos(xy) + \sin(xy)}{x^2y^2} & \text{if } xy \neq 0 \\ 0 & \text{if } xy = 0 \end{cases}$$

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**Def** (Differentiable Functions)

Let $\Omega \subseteq \mathbb{R}^n$ be an open set. Let $\vec{a} = (a_1, \dots, a_n) \in \Omega$

A fcn $f : \Omega \to \mathbb{R}$ is said to be differentiable at $\vec{a}$

if 1) $f_{x_1}(\vec{a}), \dots, f_{x_n}(\vec{a})$ all exist

2) The affine approx. $L$ of $f$ at $\vec{a}$ :

$$L(\vec{x}) := f(\vec{a}) + \sum_{i=1}^{n} f_{x_i}(\vec{a})(x_i - a_i)$$

satisfies $\displaystyle \lim_{\vec{x} \to \vec{a}} \frac{f(\vec{x}) - L(\vec{x})}{\|\vec{x} - \vec{a}\|} = 0$

error of approx. , $\varepsilon(\vec{x})$.

$$L(\vec{x}) = f(\vec{a}) + \nabla f(\vec{a}) \cdot (\vec{x} - \vec{a})$$

where $\nabla f := \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)$

is the gradient of $f$.

**Thm** If $f(\vec{x})$ is differentiable at $\vec{a}$,

then $f(\vec{x})$ is continuous at $\vec{a}$.

**Ex 2** Show that $f(x,y) = |xy|$ is differentiable at the origin.

Ans :

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h}$$
$$= \lim_{h\to 0} \frac{0-0}{h}$$
$$= 0$$

Similarly, $f_y(0,0) = 0$

Let $L(x,y) = f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0)$
$$= 0 + 0\cdot(x-0) + 0\cdot(y-0)$$
$$= 0.$$

Now
$$\frac{f(x,y)-L(x,y)}{\|(x,y)-(0,0)\|} = \frac{|xy|}{\sqrt{x^2+y^2}} = \underbrace{\frac{|x|}{\sqrt{x^2+y^2}}}_{\leq 1}|y|$$

So, $\forall (x,y) \neq (0,0)$,
$$0 \leq \frac{f(x,y)-L(x,y)}{\|(x,y)-(0,0)\|} \leq |y|$$

Since $\lim_{(x,y)\to(0,0)} |y| = 0$, it follows from Squeeze Thm that
$$\lim_{(x,y)\to(0,0)} \frac{f(x,y)-L(x,y)}{\|(x,y)-(0,0)\|} = 0.$$

Hence $f(x,y) = |xy|$ is differentiable at $(0,0)$ //

Ex 3

For each of the following functions, determine if:
(ii) the function is differentiable at the origin.

(a)

$$g(x,y) = \sqrt{x^4y^2 + y^4}$$

Ans: ii)

$$g_x(0,0) = \lim_{h\to 0} \frac{g(h,0) - g(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

$$g_y(0,0) = \lim_{k\to 0} \frac{g(0,k) - g(0,0)}{k} = \lim_{k\to 0} \frac{\sqrt{k^4} - 0}{k} = \lim_{k\to 0} k = 0$$

Let $L(x,y) = g(0,0) - g_x(0,0)\cdot(x-0) - g_y(0,0)\cdot(y-0)$
$= 0$

Then

$$0 \le \frac{g(x,y) - L(x,y)}{\|(x,y) - (0,0)\|} = \frac{\sqrt{x^4y^2+y^4} - 0}{\sqrt{x^2+y^2}} = \underbrace{\frac{|y|}{\sqrt{x^2+y^2}}}_{\le 1} \cdot \sqrt{x^4+y^2}$$

$$\le \sqrt{x^4+y^2}$$

Since $\lim_{(x,y)\to(0,0)} \sqrt{x^4+y^2} = 0$, it follows from Squeeze Thm that

$$\lim_{(x,y)\to(0,0)} \frac{g(x,y) - L(x,y)}{\|(x,y) - (0,0)\|} = 0.$$

Hence $g$ is diff. at $(0,0)$ //

(b)

$$h(x,y) = \sqrt{x^2+y^4}$$

Ans: $h_x(0,0) = \lim_{x\to 0} \frac{h(x,0)-h(0,0)}{x} = \lim_{x\to 0} \frac{\sqrt{x^2}}{x}$ ?

Note that

$$\lim_{x\to 0^+} \frac{h(x,0)-h(0,0)}{x} = \lim_{x\to 0^+} \frac{\sqrt{x^2}}{x} = \lim_{x\to 0^+} 1 = 1$$

while

$$\lim_{x\to 0^-} \frac{h(x,0)-h(0,0)}{x} = \lim_{x\to 0^-} \frac{\sqrt{x^2}}{x} = \lim_{x\to 0^-} -1 = -1$$

Hence $h_x(0,0)$ DNE.

In particular $h$ is not diff. at $(0,0)$ //

(c)

$$f(x,y) = \begin{cases} \frac{x}{x+y} & \text{if } x+y \neq 0, \\ 0 & \text{if } x+y = 0. \end{cases}$$

Ans: Along $y=0$.

$$\lim_{\substack{(x,y)\to(0,0) \\ y=0}} f(x,y) = \lim_{x\to 0} \frac{x}{x+0} = \lim_{x\to 0} 1 = 1$$

However $f(0,0) = 0$

Hence $f$ is NOT cts at $(0,0)$

So $f$ is NOT diff. at $(0,0)$ //

**Thm.** Let $f, g : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ be differentiable at $\vec{a} \in \Omega$

Then 1) $f \pm g$, $c \cdot f$, $f \cdot g$ are diff. at $\vec{a}$

2) $f/g$ is diff. at $\vec{a}$ if $g(\vec{a}) \neq 0$.

Pf: Since $f, g$ are diff at $\vec{a}$,

$$f(\vec{x}) = f(\vec{a}) + \nabla f(\vec{a}) \cdot (\vec{x} - \vec{a}) + \varepsilon_1(\vec{x}) \quad — ①$$

$$g(\vec{x}) = g(\vec{a}) + \nabla g(\vec{a}) \cdot (x - \vec{a}) + \varepsilon_2(\vec{x}) \quad — ②$$

where $\lim\limits_{\vec{x} \to \vec{a}} \dfrac{\varepsilon_1(\vec{x})}{\|\vec{x} - \vec{a}\|} = 0 \qquad \lim\limits_{\vec{x} \to \vec{a}} \dfrac{\varepsilon_2(\vec{x})}{\|\vec{x} - \vec{a}\|} = 0$

1) By 1-var product rule,

$$\frac{\partial}{\partial x_j}(f \cdot g) = f \frac{\partial g}{\partial x_j} + g \frac{\partial f}{\partial x_j}, \quad j = 1, \dots, n.$$

$$\Rightarrow \nabla(f \cdot g) = f \nabla g + g \nabla f.$$

① × ② : affine approx.

$$f(\vec{x}) g(\vec{x}) = \underline{f(\vec{a}) g(\vec{a}) + (f \nabla g + g \nabla f) \cdot (\vec{x} - \vec{a})} \qquad \lim_{\vec{x} \to \vec{a}} \frac{\mathrm{I} / \mathrm{II} / \mathrm{III} / \mathrm{IV}}{\|\vec{x} - \vec{a}\|} = 0 \, ?$$

$$\varepsilon(\vec{x}) := \left\{ \begin{array}{ll} + \; f(\vec{a})\, \varepsilon_2(\vec{x}) + g(\vec{a})\, \varepsilon_1(\vec{x}) & \mathrm{I} \; ✓ \\ + \; \nabla f(\vec{a}) \cdot (\vec{x} - \vec{a})\, \varepsilon_2(\vec{x}) + \nabla g(\vec{a}) \cdot (\vec{x} - \vec{a})\, \varepsilon_1(\vec{x}) & \mathrm{II} \; ✓ \\ + \; \varepsilon_1(\vec{x})\, \varepsilon_2(\vec{x}) & \mathrm{III} \; ✓ \\ + \; \underline{[\nabla f(\vec{a}) \cdot (\vec{x} - \vec{a})][\nabla g(\vec{a}) \cdot (\vec{x} - \vec{a})]}. & \mathrm{IV} \end{array} \right.$$

$$| \cdots | \leq \|\nabla f(\vec{a})\| \, \|\nabla g(\vec{a})\| \cdot \|\vec{x} - \vec{a}\|^2 \quad ✓$$

So $\lim\limits_{\vec{x} \to \vec{a}} \dfrac{\varepsilon(\vec{x})}{\|\vec{x} - \vec{a}\|} = 0.$

Hence $fg$ is diff. at $\vec{a}$. //