MATH 2010E TUT04

Ex 1 Determine whether the following sets are open or closed.

(a) $\mathbb{R}^n - \{x_o\}$ where x_0 is a point in $\mathbb{R}^n$.

(b) a finte set $\{x_1, \cdots, x_k\} \subset \mathbb{R}^n$.

(c) a line in $\mathbb{R}^n$.

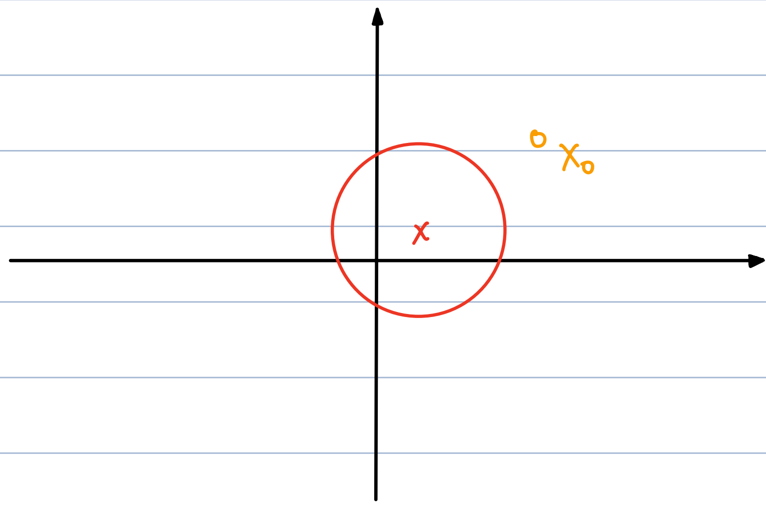
(d) $\{\frac{1}{m} | m = 1, 2, \cdots\} \subset \mathbb{R}$. What is the boundary of this set?

Recall: For $S \subseteq \mathbb{R}^n$,

1) S is open if $\forall x \in S \ \exists \varepsilon > 0$ s.t. $B_\varepsilon(x) \subseteq S$

2) S is closed if $\mathbb{R}^n \setminus S$ is open

Ans: (a) $\mathbb{R}^n - \{x_o\}$ where x_0 is a point in $\mathbb{R}^n$.

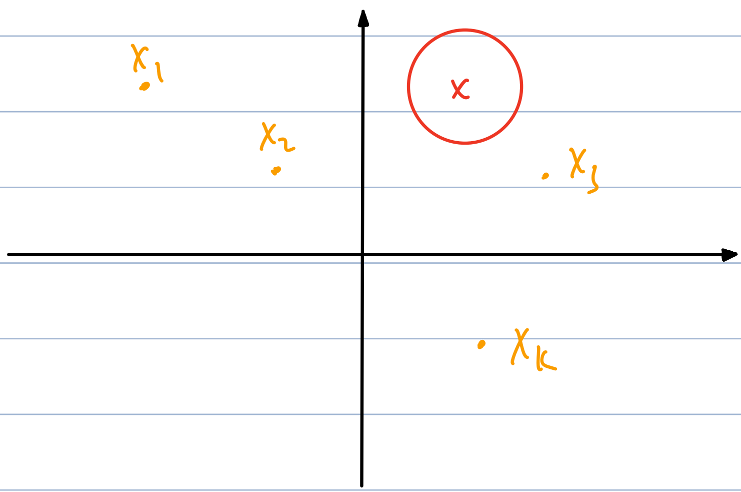


$\mathbb{R}^n \setminus \{x_0\}$ is open

$\mathbb{R}^n \setminus (\mathbb{R}^n \setminus \{x_0\}) = \{x_0\}$ contains no ball, hence NOT open

$\Rightarrow \mathbb{R}^n \setminus \{x_0\}$ is not closed

(b) a finte set $\{x_1, \cdots, x_k\} \subset \mathbb{R}^n$.



$\{x_1, \ldots, x_k\}$ is closed but not open

Ex 1 Determine whether the following sets are open or closed.

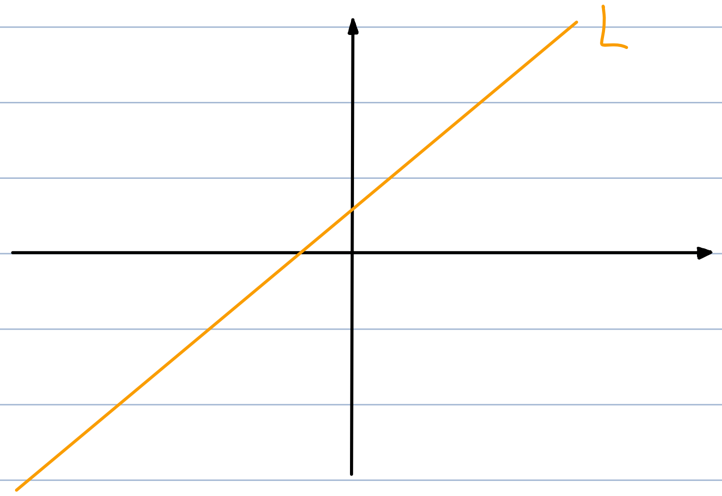
(a) $\mathbb{R}^n - \{x_o\}$ where x_0 is a point in $\mathbb{R}^n$.

(b) a finte set $\{x_1, \cdots, x_k\} \subset \mathbb{R}^n$.

(c) a line in $\mathbb{R}^n$.

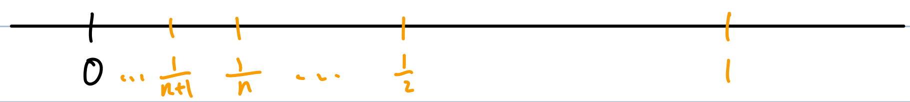
(d) $\{\frac{1}{m} | m = 1, 2, \cdots\} \subset \mathbb{R}$. What is the boundary of this set?

Ans: c)



L is closed but not open.

d)



$S_4 := \{\frac{1}{m} : m = 1, 2, \dots\} \subseteq \mathbb{R}$

is not open

and not closed. since $0 \in \mathbb{R} \setminus S_4$

but $B_\varepsilon(0) \cap S_4 \neq \phi \quad \forall \varepsilon > 0$

$\partial S_4 = \{\frac{1}{m} : m = 1, 2, \dots\} \cup \{0\}$ //

Note $x \in \partial S \iff \forall \varepsilon > 0, B_\varepsilon(x) \cap S \neq \phi$ and $B_\varepsilon(x) \cap (\mathbb{R}^n \setminus S) \neq \phi$.

Def. Let $A \subseteq \mathbb{R}^n$ and let $f: A \to \mathbb{R}$.

Suppose $\vec{a} \in \bar{A}$. $(\bar{A} = A \cup \partial A)$

We say that L is the limit of f at $\vec{a}$

$$\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = L$$

if $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\vec{x} \in B_\delta(\vec{a}) \setminus \{\vec{a}\}$

whenever $\vec{x} \in A$ and $0 < \|\vec{x} - \vec{a}\| < \delta$

we have $|f(\vec{x}) - L| < \varepsilon$

Fact: $\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = L \iff$ when $\vec{x} \to \vec{a}$ along any path limit of $f(\vec{x})$ exists and equals L

To show $\lim_{\vec{x} \to \vec{a}} f(\vec{x})$ DNE, either

- find a path along which the limit DNE
- find two paths s.t. the limits along these paths are different

Ex 2

Show from the ϵ-δ definition that

(a) $$\lim_{(x,y,z)\to(1,2,3)} x + 2y + 3z = 14;$$

Ans: a) Write $\vec{x} = (x, y, z)$, $\vec{a} = (1,2,3)$.

Note
$$\begin{aligned} |x + 2y + 3z - 14| &= |(x-1) + 2(y-2) + 3(z-3)| \\ &\leq |x-1| + 2|y-2| + 3|z-3| \\ &\leq (1+2+3)\sqrt{(x-1)^2 + (y-2)^2 + (z-3)^2} \\ &= 6\|\vec{x} - \vec{a}\| \quad < \varepsilon \end{aligned}$$

Let $\varepsilon > 0$ be given.

Take $\delta := \varepsilon/6$ (>0).

Now, if $0 < \|\vec{x} - \vec{a}\| < \delta$, then
$$\begin{aligned} |x + 2y + 3z - 14| &\leq 6\|\vec{x} - \vec{a}\| \\ &< 6\delta \\ &= \varepsilon \end{aligned}$$

Hence $\lim_{\vec{x}\to\vec{a}} (x + 2y + 3z) = 14$. //

Ex 3

Find the following limits or explain why the limits do not exist.

(a) $\displaystyle\lim_{(x,y)\to(1,-1)} \frac{x^2 - xy - 2y^2}{x+y}$

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y}{x^3+y}$

(c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^6}{x^4+y^2}$

Ans: a) $\displaystyle\lim_{(x,y)\to(1,-1)} \frac{x^2-xy-2y^2}{x+y} = \lim_{(x,y)\to(1,-1)} \frac{(x+y)(x-2y)}{x+y}$

$\displaystyle = \lim_{(x,y)\to(1,-1)} (x-2y)$

$= (1) - 2(-1) = 3$

b) Along $y=0$

$\displaystyle\lim_{\substack{(x,y)\to(0,0)\\ y=0}} \frac{y}{x^3+y} = \lim_{x\to 0} 0 = 0$

Along $x=0$

$\displaystyle\lim_{\substack{(x,y)\to(0,0)\\ x=0}} \frac{y}{x^3+y} = \lim_{y\to 0} \frac{y}{0^3+y} = \lim_{y\to 0} 1 = 1$

So different limits along different paths

$\displaystyle\Rightarrow \lim_{(x,y)\to(0,0)} \frac{y}{x^3+y}$ DNE

(c) $\lim_{(x,y)\to(0,0)} \frac{x^6}{x^4+y^2}$

Ans: c) Note that, $\forall\ (x,y) \neq (0,0)$,

$$\frac{x^6}{x^4+y^2} = \underbrace{\frac{x^4}{x^4+y^2}}_{\leq 1} \cdot \underbrace{x^2}_{\text{small}}$$

$$\Rightarrow \quad 0 \leq \frac{x^6}{x^4+y^2} \leq x^2$$

Since $\lim_{(x,y)\to(0,0)} x^2 = 0$,

by Sandwich Thm, we conclude that

$$\lim_{(x,y)\to(0,0)} \frac{x^6}{x^4+y^2} = 0$$

(Method II) Using polar coordinates,

$$\lim_{(x,y)\to(0,0)} \frac{x^6}{x^4+y^2} = \lim_{r\to 0^+} \frac{(r\cos\theta)^6}{(r\cos\theta)^4+(r\sin\theta)^2}$$

$$= \lim_{r\to 0^+} \frac{r^4\cos^6\theta}{r^2\cos^4\theta+\sin^2\theta}$$

$$= 0 \qquad \text{by squeeze thm}$$

Since $0 \leq \underbrace{\frac{r^2\cos^4\theta}{r^2\cos^4\theta+\sin^2\theta}}_{\leq 1} \cdot \underbrace{r^2\cos^2\theta}_{\leq r^2} \leq r^2$

Ex 4

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^4} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Evaluate $\lim_{(x,y)\to(0,0)} f(x,y)$ if it exists.

(Try polar coordinates)

In polar coordinates, the fcn becomes,

$$f(r\cos\theta, r\sin\theta) = \frac{(r\cos\theta)(r\sin\theta)^2}{(r\cos\theta)^2 + (r\sin\theta)^4}$$

$$= \frac{r\cos\theta\sin^2\theta}{\cos^2\theta + r^2\sin^4\theta}$$

$$\Rightarrow \lim_{r\to0^+} f(r\cos\theta, r\sin\theta) = \begin{cases} 0 & \text{if } \cos\theta = 0 \\ \dfrac{0}{\cos^2\theta + 0} = 0 & \text{if } \cos\theta \neq 0 \end{cases}$$

$$= 0 \quad \text{for any } \theta$$

Can we conclude that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$?

NO!

$\lim_{r\to0^+} f(r\cos\theta, r\sin\theta) = 0 \;\; \forall\, \theta$ is not enough.

Ex 4

Let $f:\mathbb{R}^2\to\mathbb{R}$ be defined by

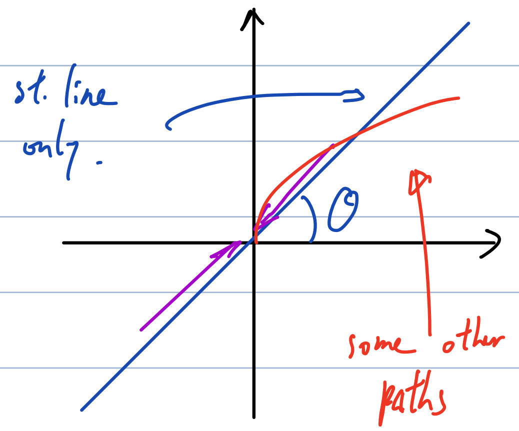
$$f(x,y)=\begin{cases}\dfrac{xy^2}{x^2+y^4} & \text{if } (x,y)\neq(0,0);\\ 0 & \text{if } (x,y)=(0,0).\end{cases}$$

In fact, along the path $x=y^2$,

$$f(y^2,y^2)=\frac{y^2y^2}{(y^2)^2+y^4}=\frac{1}{2}$$

$$\Rightarrow \lim_{\substack{(x,y)\to(0,0)\\ x=y^2}} f(x,y)=\frac{1}{2}\neq 0$$

So $\lim_{(x,y)\to(0,0)} f(x,y)$ DNE.



Ex 5 Find

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2+y^2+z^2}$$

if it exists. (Hint: spherical coordinates)

Ans: Using spherical coordinates,

$$\frac{xyz}{x^2+y^2+z^2} = \frac{(\rho\sin\phi\cos\theta)(\rho\sin\phi\sin\theta)(\rho\cos\phi)}{\rho^2}$$

$$= \rho\cos\phi\sin^2\phi\cos\theta\sin\theta$$

Note $|\rho\cos\phi\sin^2\phi\cos\theta\sin\theta| = \rho|\cos\phi||\sin\phi|^2|\cos\theta||\sin\theta| \le \rho$

and $\lim\limits_{\rho\to 0} \rho = 0$

By the squeeze thm, $\lim\limits_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2+y^2+z^2} = 0$ //

Alternative sol:

Note $|xy| \le 2|x||y| \le x^2+y^2$

Hence $$\left|\frac{xyz}{x^2+y^2+z^2}\right| \le \frac{x^2+y^2}{x^2+y^2+z^2}|z|$$

$$\le |z|.$$

Since $\lim\limits_{(x,y,z)\to(0,0,0)} |z| = 0$, it follows from the squeeze thm that

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2+y^2+z^2} = 0$$ //