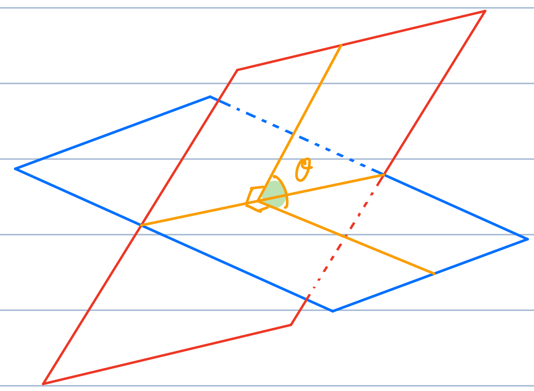
MATH 2010 E TUT 03

Ex 1 Find the angle between the two planes

$$P_1 : x + 2y + 3z = 1, P_2 : -2x + 3y - z = 3.$$

Ans: $\angle$ between P_1, P_2

$= \angle$ between their normals $\vec{n}_1, \vec{n}_2$



Note $\vec{n}_1 = (1, 2, 3)$

$\vec{n}_2 = (-2, 3, -1)$

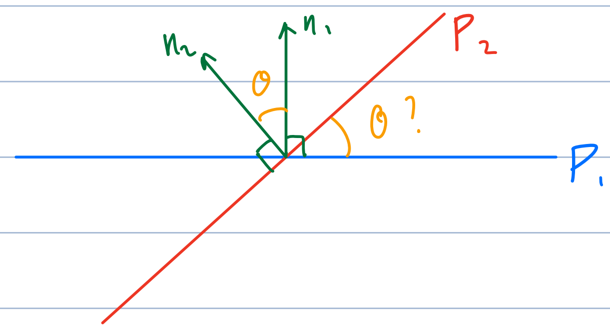
So $\angle$ between P_1, P_2

$= \cos^{-1}\left(\dfrac{\vec{n}_1 \cdot \vec{n}_2}{\|\vec{n}_1\| \, \|\vec{n}_2\|}\right)$

$= \cos^{-1}\left(\dfrac{-2 + 6 - 3}{\sqrt{1^2+2^2+3^2} \cdot \sqrt{(-2)^2+3^2+(-1)^2}}\right)$

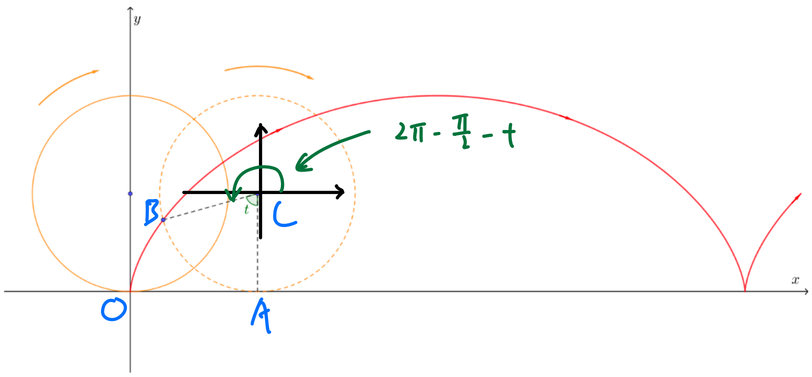
$= \cos^{-1}\left(\dfrac{1}{14}\right)$

$\approx 85.90^\circ$



Ex 2

In the following diagram, a circular disk of radius 1 in the plane xy rolls without slipping along the x-axis and the curve is the locus of a fixed point on the circumference which is called a *cycloid*.



(a) Give a parametrization of the cycloid.

(b) Find the arc length of the cycloid corresponding to a complete rotation of the disk.

https://www.desmos.com/calculator/3qsidj1kq5

Ans: a) When $\angle ACB = t$,

$OA = \overset{\frown}{BA} = (1)(t) = t$

$AC = 1$

$\Rightarrow \overrightarrow{OC} = (t, 1)$

$\overrightarrow{CB} = \left(\cos\left(2\pi - \frac{\pi}{2} - t\right), \sin\left(2\pi - \frac{\pi}{2} - t\right)\right)$

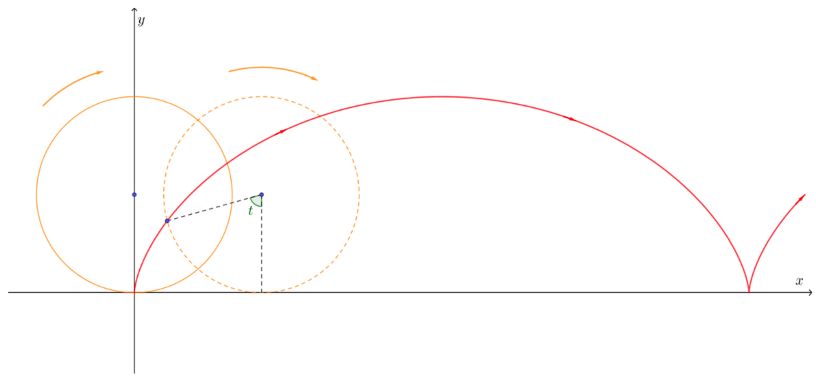
$= (-\sin t, -\cos t)$

Hence, a parametrization of the cycloid is given by

$\vec{r}(t) = \overrightarrow{OB} = \overrightarrow{OC} + \overrightarrow{CB}$

$= (t - \sin t, 1 - \cos t), \quad t \in \mathbb{R}$

In the following diagram, a circular disk of radius 1 in the plane xy rolls without slipping along the x-axis and the curve is the locus of a fixed point on the circumference which is called a *cycloid*.



(a) Give a parametrization of the cycloid.

(b) Find the arc length of the cycloid corresponding to a complete rotation of the disk.

Ans: b)
$$\vec{r}(t) = (t - \sin t,\ 1 - \cos t), \quad t \in \mathbb{R}$$
$$\vec{r}\,'(t) = (1 - \cos t,\ \sin t), \quad t \in \mathbb{R}$$
$$\Rightarrow \|\vec{r}\,'(t)\|^2 = (1-\cos t)^2 + \sin^2 t$$
$$= 1 - 2\cos t + \cos^2 t + \sin^2 t$$
$$= 2 - 2\cos t$$

Arc length of cycloid corr. to a complete rotation of the disk

$$= \int_0^{2\pi} \|\vec{r}\,'(t)\|\, dt$$
$$= \int_0^{2\pi} \sqrt{2-2\cos t}\, dt \qquad \cos 2\theta = 1 - 2\sin^2\theta$$
$$= \int_0^{2\pi} \sqrt{4\sin^2 \tfrac{t}{2}}\, dt \qquad \Rightarrow 1 - \cos 2\theta = 2\sin^2\theta$$
$$= \int_0^{2\pi} 2\sin\tfrac{t}{2}\, dt \qquad \text{since } \sin\tfrac{t}{2} \geq 0 \quad \forall\, t \in [0, 2\pi]$$
$$= -4\cos\tfrac{t}{2}\Big|_0^{2\pi}$$
$$= -4(-1 - (-1)) = 8$$

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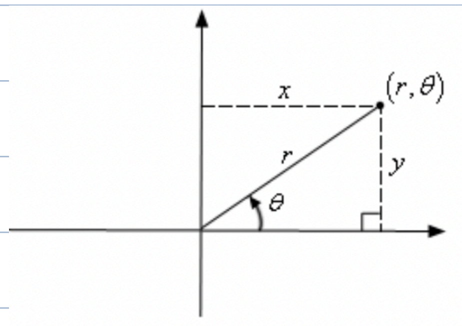
Polar coordinates

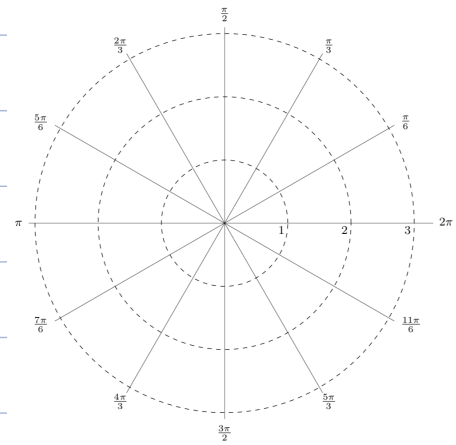
A point $P = (x, y) \in \mathbb{R}^2$ can be represented by (r, θ), where

$r =$ distance from the origin

$\theta =$ angle from the x-axis to $\overrightarrow{OP}$ in counter clockwise direction

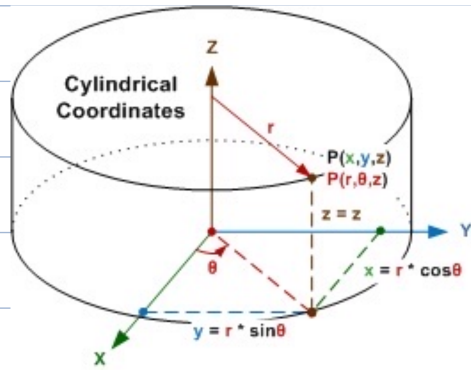


$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \longleftrightarrow \begin{cases} r = \sqrt{x^2+y^2} \\ \tan\theta = \frac{y}{x} \text{ and Quadrant.} \end{cases}$$



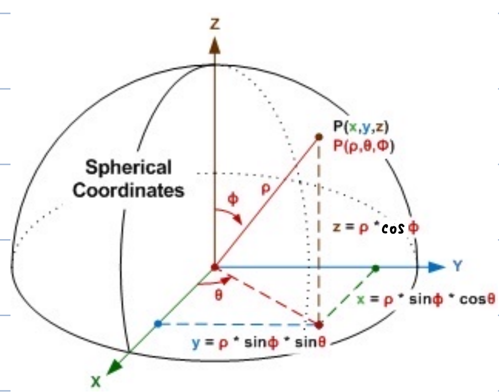
Cylindrical Coordinates (r, θ, z)

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$



Spherical Coordinates (ρ, θ, φ)

$$\begin{cases} x = \rho\sin\varphi\cos\theta \\ y = \rho\sin\varphi\sin\theta \\ z = \rho\cos\varphi \end{cases}$$



radial distance $\rho \geq 0$

azimuthal angle $\theta \in [0, 2\pi)$ or $[-\pi, \pi)$

polar angle $\varphi \in [0, \pi]$

Ex 3 (Arclength in polar coordinates) Let $r = r(\theta)$ be a curve in $\mathbb{R}^2$ in polar coordinates. Show that the arclenth of the curve from θ_1 to θ_2 is given by

$$\int_{\theta_1}^{\theta_2} \sqrt{(\frac{dr}{d\theta})^2 + r^2 d\theta}.$$

Ans: In polar cordinates, $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

So the curve is parametrized by

$\vec{x}(\theta) = (r(\theta)\cos\theta,\ r(\theta)\sin\theta)$, $\theta_1 \le \theta \le \theta_2$

Then $\vec{x}'(\theta) = (r'(\theta)\cos\theta - r(\theta)\sin\theta,\ r'(\theta)\sin\theta + r(\theta)\cos\theta)$

$$\|\vec{x}'(\theta)\|^2 = r'^2\cos^2\theta - \cancel{2r'r\cos\theta\sin\theta} + r^2\sin^2\theta + r'^2\sin^2\theta + \cancel{2r'r\cos\theta\sin\theta} + r^2\cos^2\theta$$

$$= r'^2 + r^2$$

$$\Rightarrow \|\vec{x}'(\theta)\| = \sqrt{(\frac{dr}{d\theta})^2 + r^2}$$

Hence, arclength $= \int_{\theta_1}^{\theta_2} \|\vec{x}'(\theta)\| d\theta$

$$= \int_{\theta_1}^{\theta_2} \sqrt{(\frac{dr}{d\theta})^2 + r^2}\, d\theta$$

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Ex 4 Find the arclength of the curve: $r(\theta) = a(1 - cos\theta)$ where $a > 0$ is a constant and $0 \le \theta \le 2\pi$.

Ans:
$$r(\theta) = a(1 - \cos\theta)$$
$$r'(\theta) = a\sin\theta$$

$$\int_0 (r')^2 + r^2 = a^2\sin^2\theta + a^2(1 - 2\cos\theta + \cos^2\theta)$$
$$= 2a^2(1 - \cos\theta)$$
$$= 4a^2\sin^2\frac{\theta}{2}$$

By Ex 3, arclength $= \int_0^{2\pi} \sqrt{(r')^2 + r^2}\, d\theta$
$$= \int_0^{2\pi} 2a\sin\frac{\theta}{2}\, d\theta \qquad \sin\frac{\theta}{2} \geq 0 \ \forall \theta \in [0, 2\pi].$$
$$= -4a\cos\frac{\theta}{2}\Big|_0^{2\pi}$$
$$= -4a(-1 - 1)$$
$$= 8a$$

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Ex 5. Describe the following curves in polar coordinates:

a) $r = 2\cos\theta - 3\sin\theta$

b) $r^2 \sin 2\theta = 1$

Ans: a) $r^2 = 2r\cos\theta - 3r\sin\theta$

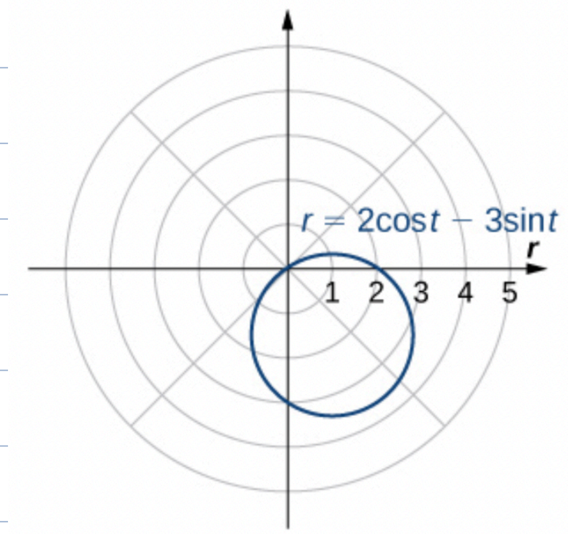
$x^2 + y^2 = 2x - 3y$

$(x^2 - 2x) + (y^2 + 3y) = 0$

$(x-1)^2 - 1 + \left(y + \frac{3}{2}\right)^2 - \frac{9}{4} = 0$

$(x-1)^2 + \left(y + \frac{3}{2}\right)^2 = \frac{13}{4}$

a circle

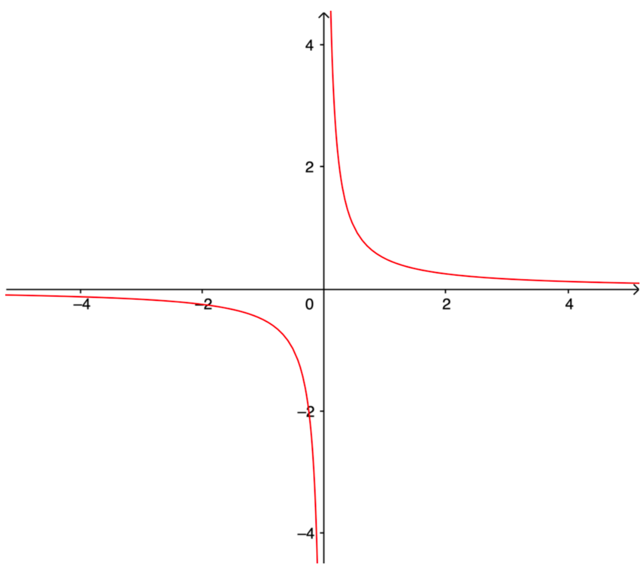


b) $r^2 \sin 2\theta = 1$

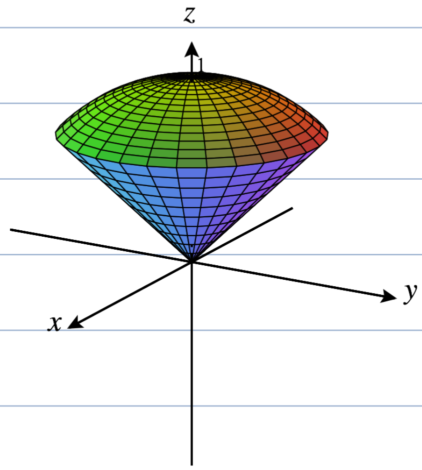
$r^2 (2\sin\theta\cos\theta) = 1$

$2xy = 1$

a hyperbola

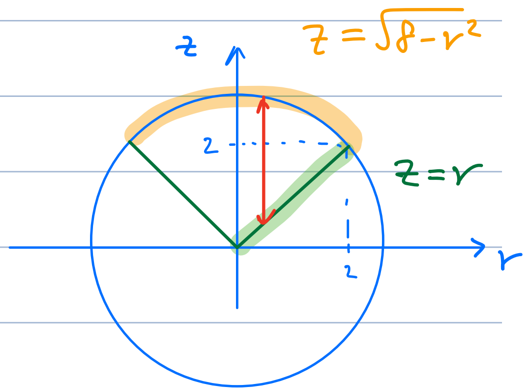


Ex 6 Let $S \subseteq \mathbb{R}^3$ be the solid bounded by the cone $z = \sqrt{x^2+y^2}$, and the hemisphere $z = \sqrt{8-x^2-y^2}$.
Express S in a) cylindrical coordinates
b) spherical coordinates



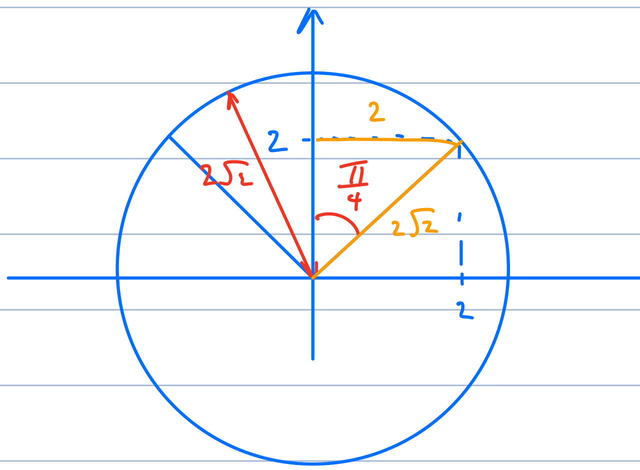
Ans: a) When the cone and the hemisphere meet

$$\sqrt{x^2+y^2} = \sqrt{8-x^2-y^2}$$
$$r = \sqrt{8-r^2}$$
$$2r^2 = 8$$
$$r = 2$$



So S: $0 \le \theta \le 2\pi$, $0 \le r \le 2$, $r \le z \le \sqrt{8-r^2}$

b)



$$\sin\phi = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}}$$
$$\phi = \frac{\pi}{4}$$

So S: $0 \le \theta \le 2\pi$, $0 \le \rho \le 2\sqrt{2}$, $0 \le \phi \le \frac{\pi}{4}$ //