MATH 2010E TUT 02

Cross Product (defined only in $\mathbb{R}^3$)

For $\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3) \in \mathbb{R}^3$, define

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \qquad \text{(determinant)}$$

$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \hat{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \hat{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \hat{k}$$

$$= (a_2 b_3 - a_3 b_2)\hat{i} - (a_1 b_3 - a_3 b_1)\hat{j} + (a_1 b_2 - a_2 b_1)\hat{k}$$

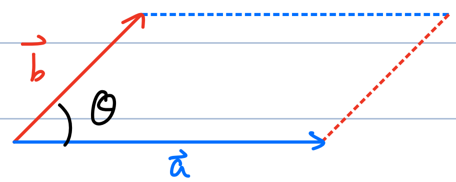
Then

- $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$

  $(\alpha \vec{a} + \beta \vec{b}) \times \vec{c} = \alpha(\vec{a} \times \vec{c}) + \beta(\vec{b} \times \vec{c})$, $\alpha, \beta \in \mathbb{R}$

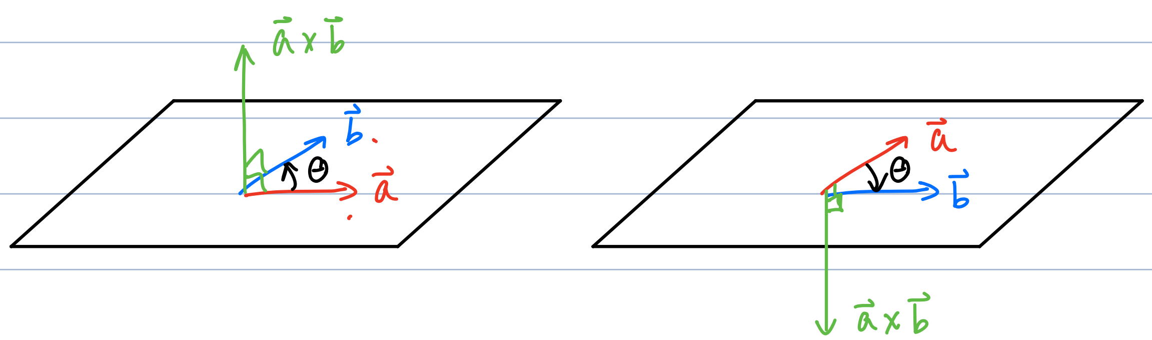
- $\| \vec{a} \times \vec{b} \| = \|\vec{a}\| \, \|\vec{b}\| \sin\theta$

  $=$ area of parallelogram spanned by $\vec{a}, \vec{b}$.



$0 \le \theta \le \pi$

- $\vec{a} \times \vec{b} = \vec{0} \iff \vec{a} = r\vec{b}$ or $\vec{b} = r\vec{a}$ for some $r \in \mathbb{R}$
- $\vec{a} \times \vec{b} \perp \vec{a}, \vec{b}$
- Right Hand Rule :



Ex 1 Find the length and direction (when defined) of $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$, where $\mathbf{u} = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}$, $\mathbf{v} = \mathbf{i} - \mathbf{k}$.

Ans:

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -2 & -1 \\ 1 & 0 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} -2 & -1 \\ 0 & -1 \end{vmatrix} \hat{i} - \begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix} \hat{j} + \begin{vmatrix} 2 & -2 \\ 1 & 0 \end{vmatrix} \hat{k}$$

$$= 2\hat{i} + \hat{j} + 2\hat{k}$$

So length $= \|\vec{u} \times \vec{v}\| = \sqrt{2^2 + 1^2 + 2^2} = 3$

direction $= \dfrac{\vec{u} \times \vec{v}}{\|\vec{u} \times \vec{v}\|} = \dfrac{1}{3}(2\hat{i} + \hat{j} + 2\hat{k})$

Since $\vec{v} \times \vec{u} = -(\vec{u} \times \vec{v})$, so

length $= 3$

direction $= -\dfrac{1}{3}(2\hat{i} + \hat{j} + 2\hat{k})$ //

Ex 2 **Cancelation in cross products** If $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w}$ and $\mathbf{u} \neq \mathbf{0}$, then does $\mathbf{v} = \mathbf{w}$? Give reasons for your answer.

Ans: Consider $\hat{i} \times \hat{i} = \vec{0}$

No.

Take $\vec{u} = \hat{i} \neq \vec{0}$

$\vec{v} = 2\hat{i}$, $\vec{w} = \hat{i}$

Then $\vec{v} \neq \vec{w}$

but $\vec{u} \times \vec{v} = \vec{0} = \vec{u} \times \vec{w}$. //

Ex 3 **Double cancelation** If $\mathbf{u} \neq \mathbf{0}$ and if $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w}$ and $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$, then does $\mathbf{v} = \mathbf{w}$? Give reasons for your answer.

Ans: Yes :

$$\begin{cases} \vec{u} \times (\vec{v} - \vec{w}) = \vec{0} & ① \\ \vec{u} \cdot (\vec{v} - \vec{w}) = 0 & ② \end{cases}$$

① $\Rightarrow$ $\vec{v} - \vec{w} = r\vec{u}$ for some $r \in \mathbb{R}$ (since $\vec{u} \neq \vec{0}$)

By ②, $\vec{u} \cdot (r\vec{u}) = 0$

$\Rightarrow$ $r\|\vec{u}\|^2 = 0$

$\Rightarrow$ $r = 0$ (since $\vec{u} \neq \vec{0}$)

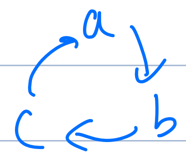
Hence $\vec{v} = \vec{w}$ //

(Scalar) Triple Product

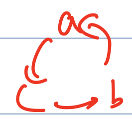
The triple product of $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$ is defined to be

$$(\vec{a}\times\vec{b})\cdot\vec{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \leftarrow \text{scalar}$$

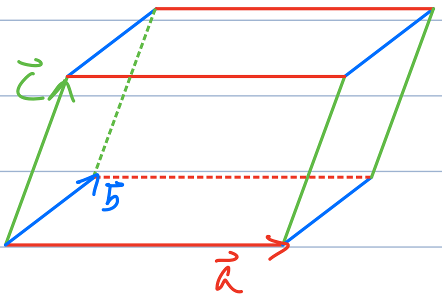
- $(\vec{a}\times\vec{b})\cdot\vec{c} = (\vec{b}\times\vec{c})\cdot\vec{a} = (\vec{c}\times\vec{a})\cdot\vec{b}$



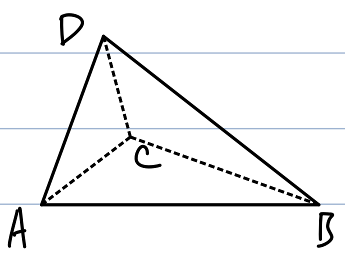
$= -(\vec{b}\times\vec{a})\cdot\vec{c} = -(\vec{c}\times\vec{b})\cdot\vec{a} = -(\vec{a}\times\vec{c})\cdot\vec{b}$



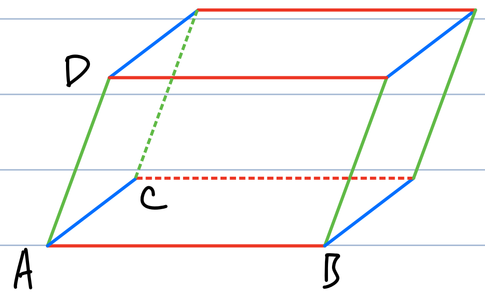
- $|(\vec{a}\times\vec{b})\cdot\vec{c}|$ = Volume of parallelepipe spanned by $\vec{a}, \vec{b}, \vec{c}$



Ex4. Let $A=(1,0,1)$, $B=(1,1,2)$
$C=(2,1,1)$, $D=(2,1,3)$.
Find the volume of tetrahedron $ABCD$



Tetrahedron



Parallelepiped

Ans: Volume of tetrahedron

$$= \frac{1}{3}\cdot \text{Area}(\triangle ABC)\cdot \text{height}$$

$$= \frac{1}{3}\cdot\frac{1}{2}\,\text{Area}(▱_{A\,B}^{\;C})\cdot \text{height}$$

$$= \frac{1}{6}\cdot \text{Volume of parallelepiped}$$

$$= \frac{1}{6}\left|(\vec{AB}\times\vec{AC})\cdot\vec{AD}\right|$$

Now $\vec{AB} = (1,1,2)-(1,0,1) = (0,1,1)$

$\vec{AC} = (1,0,-1)$

$\vec{AD} = (1,1,2)$

$A=(1,0,1)$, $B=(1,1,2)$
$C=(2,1,1)$, $D=(2,1,3)$.

$$\Rightarrow (\vec{AB}\times\vec{AC})\cdot\vec{AD} = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & 2 \end{vmatrix}$$

$$= 0 - (1)\begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} + (1)\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}$$

$$= 0 - (3) + 1 \quad = -2$$

So volume of tetrahedron $= \frac{1}{6}|-2| = \frac{1}{3}$ //

Ex 5 **Triangle area** Find a concise 3×3 determinant formula that gives the area of a triangle in the xy-plane having vertices (a_1, a_2), (b_1, b_2), and (c_1, c_2).

Ans: Put the vertices on the plane $z = 1$:

$A = (a_1, a_2, 1)$, $B = (b_1, b_2, 1)$, $C = (c_1, c_2, 1)$

Let $O = (0, 0, 0)$.

Then volume of tetrahedron $OABC = \frac{1}{3}(\text{Area}(\triangle)) \cdot \text{height}$

$$\Rightarrow \quad \frac{1}{6}\left|\begin{vmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ c_1 & c_2 & 1 \end{vmatrix}\right| = \frac{1}{3}(\text{Area}(\triangle)) \cdot 1$$

absolute value.

$$\Rightarrow \quad \text{Area}(\triangle) = \frac{1}{2}\left|\begin{vmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ c_1 & c_2 & 1 \end{vmatrix}\right|$$

//

Ex 6 Show that 4 points $A=(2,0,0), B=(3,2,-3), C=(1,-1,2), D=(7,-1,-4)$ are coplanar.

Ans: Note

A, B, C, D are coplanar $\Leftrightarrow$ $\vec{AB}, \vec{AC}, \vec{AD}$ linearly dependent

$$\Leftrightarrow \begin{vmatrix} \vec{AB} \\ \vec{AC} \\ \vec{AD} \end{vmatrix} = (\vec{AB} \times \vec{AC}) \cdot \vec{AD} = 0$$

$$\vec{AB} = (3,2,-3) - (2,0,0) = (1,2,-3)$$
$$\vec{AC} = (1,-1,2) - (2,0,0) = (-1,-1,2)$$
$$\vec{AD} = (7,-1,-4) - (2,0,0) = (5,-1,-4)$$

$$(\vec{AB} \times \vec{AC}) \cdot \vec{AD} = \begin{vmatrix} 1 & 2 & -3 \\ -1 & -1 & 2 \\ 5 & -1 & -4 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2 & -3 \\ 0 & 1 & -1 \\ 0 & -11 & 11 \end{vmatrix} \qquad \begin{matrix} R_2 + R_1 \to R_2 \\ R_3 - 5R_1 \to R_3 \end{matrix}$$

$$= \begin{vmatrix} 1 & 2 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{vmatrix} \qquad R_3 + 11R_2 \to R_3$$

$$= (1)(1)(0) \qquad \text{(triangular matrix)}$$

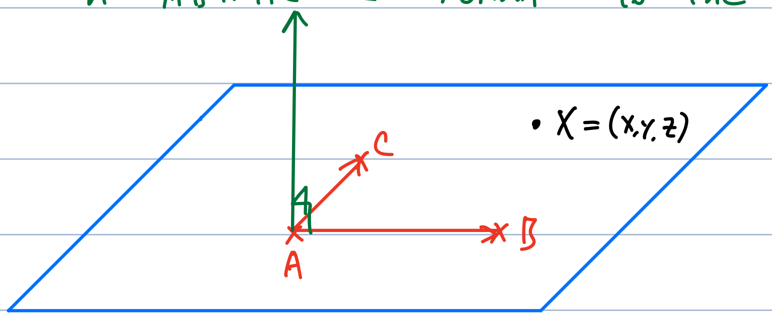
$$= 0$$

So A, B, C, D are coplanar //

Ex 7 Find an equation of the plane through $(1,1,-1)$, $(2,0,2)$ and $(0,-2,1)$.

A B C

$\vec{n} := \vec{AB} \times \vec{AC}$ ← normal to the plane



Now if $X=(x,y,z)$ is an arbitrary pt. on the plane,

then $\vec{AX} \perp \vec{n}$, i.e.

$$\vec{AX} \cdot \vec{n} = 0.$$ plane equation

Ans: $\vec{AB} = (1,-1,3)$, $\vec{AC} = (-1,-3,2)$

A normal to the plane is

$$\vec{n} := \vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 3 \\ -1 & -3 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 3 \\ -3 & 2 \end{vmatrix} \hat{i} - \begin{vmatrix} 1 & 3 \\ -1 & 2 \end{vmatrix} \hat{j} + \begin{vmatrix} 1 & -1 \\ -1 & -3 \end{vmatrix} \hat{k}$$

$$= 7\hat{i} - 5\hat{j} - 4\hat{k}$$

Hence, the eqn of the plane is

$$[(x,y,z) - (1,1,-1)] \cdot (7,-5,-4) = 0$$

$$7(x-1) - 5(y-1) - 4(z+1) = 0$$

$$7x - 5y - 4z = 7 - 5 + 4 = 6$$

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