MATH 2010E TUTO 1

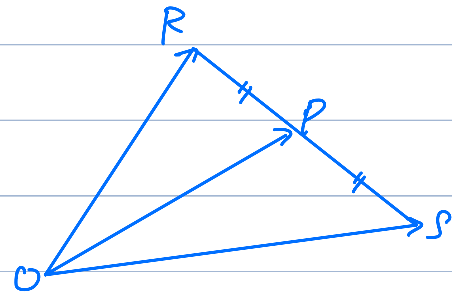
Ex 1

Find the following vectors:

(i) The vector $\overrightarrow{OP}$ where O is the origin and P is the midpoint of segment RS, where $R = (2, -1)$ and $S = (-4, 3)$

(ii) The sum of $\overrightarrow{AB}$ and $\overrightarrow{CD}$, where $A = (1, -1)$, $B = (2, 0)$, $C = (-1, 3)$, and $D = (-2, 2)$

(iii) The unit vectors making an angle $\theta = 2\pi/3$ with the positive x-axis



Ans: i) $\overrightarrow{OP} = \frac{1}{2}(\overrightarrow{OR} + \overrightarrow{OS})$

$= \frac{1}{2}(2-4, -1+3)$

$= (-1, 1)$

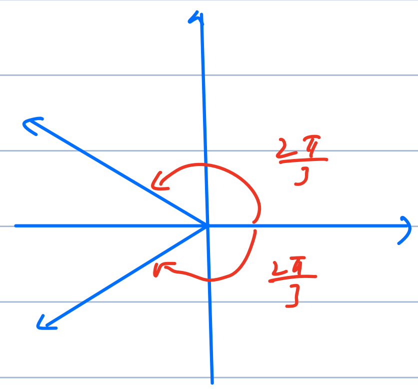
ii) $\overrightarrow{AB} = (2,0) - (1,-1) = (1, 1)$

$\overrightarrow{CD} = (-2,2) - (-1,3) = (-1,-1)$

So $\overrightarrow{AB} + \overrightarrow{CD} = (1,1) + (-1,1) = (0,0)$

iii) $\mathbb{R}^2$: $\left(\cos\frac{2\pi}{3}, \sin\frac{2\pi}{3}\right) = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

or $\left(\cos\frac{-2\pi}{3}, \sin\frac{-2\pi}{3}\right) = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$



$\mathbb{R}^n, n \geq 3$: Let $\vec{u} = (a_1, a_2, \dots, a_n)$ be a unit vector

Then $\cos\frac{2\pi}{3} = \frac{\vec{u}\cdot(1,0,\dots,0)}{\|\vec{u}\|\,\|(1,0,\dots,0)\|}$

$\Leftrightarrow \quad -\frac{1}{2} = \frac{a_1}{1}$

$\Leftrightarrow \quad a_1 = -\frac{1}{2}$

Also, $a_1^2 + a_2^2 + \cdots + a_n^2 = 1$

$\Leftrightarrow \quad a_2^2 + \cdots + a_n^2 = \frac{3}{4}$

The required vectors are

$(a_1, \dots, a_n)$, where $a_1 = -\frac{1}{2}$

$a_2^2 + \cdots + a_n^2 = \frac{3}{4}$

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Ex2 Compute $\vec{u}+\vec{v}$, $\vec{u}-\vec{v}$, $\vec{u}\cdot\vec{v}$, $||\vec{u}||$, $||\vec{v}||$, and the cosine of the angle between $\vec{u}$ and $\vec{v}$ for $\vec{u}=(2,1,0)$ and $\vec{v}=(1,3,-1)$.

Ans:

$\vec{u} = (2,1,0)$, $\vec{v} = (1,3,-1) \in \mathbb{R}^3$

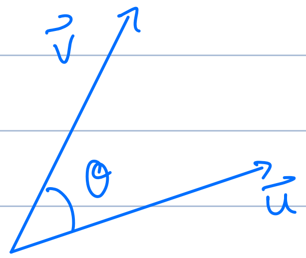
$$\vec{u}+\vec{v} = (2+1,\ 1+3,\ 0+(-1)) = (3,4,-1) \in \mathbb{R}^3$$

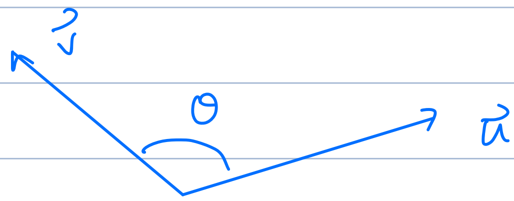
$$\vec{u}-\vec{v} = (2-1,\ 1-3,\ 0-(-1)) = (1,-2,1) \in \mathbb{R}^3$$

$$\vec{u}\cdot\vec{v} = 2\cdot 1 + 1\cdot 3 + 0(-1) = 5 \quad \leftarrow \text{scalar}$$

$$||\vec{u}|| = \sqrt{\vec{u}\cdot\vec{u}} = \sqrt{2^2+1^2+0^2} = \sqrt{5}$$

$$||\vec{v}|| = \sqrt{\vec{v}\cdot\vec{v}} = \sqrt{1^2+3^2+(-1)^2} = \sqrt{11}$$





$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{||\vec{u}||\ ||\vec{v}||} = \frac{5}{\sqrt{5}\ \sqrt{11}} = \sqrt{\frac{5}{11}} \quad //$$

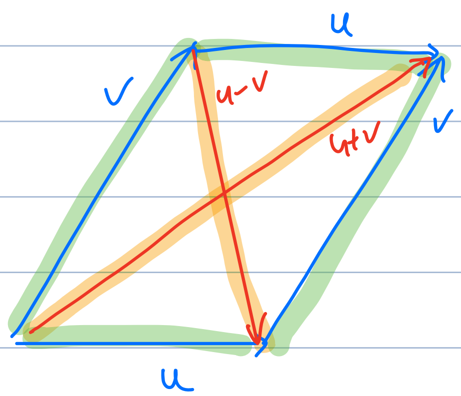
Ex 3 (Parallelogram law) Let $\vec{u}, \vec{v} \in \mathbb{R}^n$. Prove that $||\vec{u}+\vec{v}||^2 + ||\vec{u}-\vec{v}||^2 = 2(||\vec{u}||^2 + ||\vec{v}||^2)$.

Ans:

$$\begin{aligned}\|u+v\|^2 &= (u+v)\cdot(u+v) \\ &= u\cdot(u+v) + v\cdot(u+v) \\ &= u\cdot u + \cancel{u\cdot v} + \cancel{v\cdot u} + v\cdot v\end{aligned}$$

$$\begin{aligned}\|u-v\|^2 &= (u-v)\cdot(u-v) \\ &= u\cdot(u-v) - v\cdot(u-v) \\ &= u\cdot u - \cancel{u\cdot v} - \cancel{v\cdot u} + v\cdot v\end{aligned}$$

So $$\begin{aligned}\|u+v\|^2 + \|u-v\|^2 &= 2\,u\cdot u + 2\,v\cdot v \\ &= 2\left(\|u\|^2 + \|v\|^2\right)\end{aligned}$$



Ex4 Show that $3(a^2+b^2+c^2) \geq (a+b+c)^2$ for any $a, b, c \in \mathbb{R}$.

__Recall__ : (Cauchy - Schwarz Inequality) $\left(\sum_{i=1}^{n} x_i y_i\right)^2 \leq \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right)$

Let $\vec{x}, \vec{y} \in \mathbb{R}^n$. Then

$$|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \, \|\vec{y}\| .$$

Moreover, "=" holds iff

$\vec{x} = r\vec{y}$ or $\vec{y} = r\vec{x}$ for some $r \in \mathbb{R}$

Ans: Consider $\vec{x} = (a, b, c)$

$\vec{y} = (1, 1, 1)$

By Cauchy - Schwarz Inequality,

$$|\vec{x} \cdot \vec{y}|^2 \leq \|\vec{x}\|^2 \|\vec{y}\|^2$$

i.e. $$(a \cdot 1 + b \cdot 1 + c \cdot 1)^2 \leq (a^2 + b^2 + c^2)(1^2 + 1^2 + 1^2)$$

$$(a+b+c)^2 \leq 3(a^2+b^2+c^2)$$

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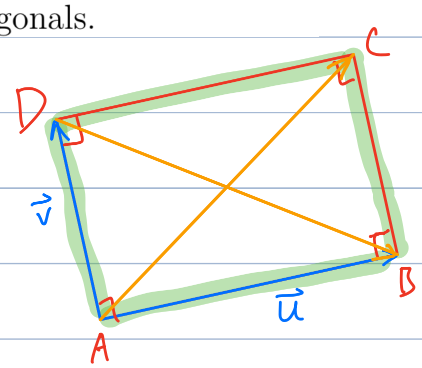
Ex 5 Show that squares are the only rectangles with perpendicular diagonals.



Ans: Consider a rectangle ABCD

Let $\vec{AB} = \vec{u}$

$\vec{AD} = \vec{v}$

Then $\vec{AC} = \vec{AB} + \vec{BC} = \vec{u} + \vec{v}$

$\vec{DB} = \vec{DA} + \vec{AB} = -\vec{v} + \vec{u} = \vec{u} - \vec{v}$

Suppose $\vec{AC} \perp \vec{DB}$

Then $\vec{AC} \cdot \vec{DB} = 0$

$(\vec{u} + \vec{v}) \cdot (\vec{u} - \vec{v}) = 0 \quad = \cancel{\vec{u} \cdot \vec{v}}$

$\vec{u} \cdot \vec{u} - \cancel{\vec{u} \cdot \vec{v}} + \vec{v} \cdot \vec{u} - \vec{v} \cdot \vec{v} = 0$

So $\vec{u} \cdot \vec{u} = \vec{v} \cdot \vec{v}$

i.e. $\|\vec{u}\|^2 = \|\vec{v}\|^2$

$\Rightarrow \|\vec{u}\| = \|\vec{v}\|$

Since $\vec{u} \perp \vec{v}$ and $\|\vec{u}\| = \|\vec{v}\|$,

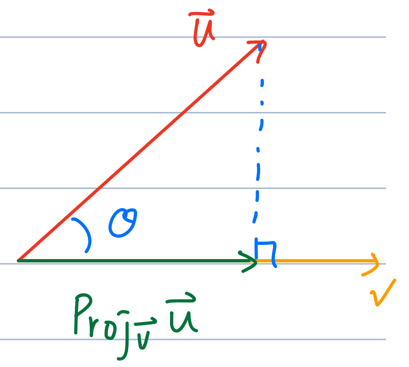
ABCD is a square //

Ex6 a) Let $\mathbf{u}, \mathbf{v}$ be two vectors in $\mathbb{R}^3$. We denote by

$$\mathrm{Proj}_{\mathbf{v}}\mathbf{u} := \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v} \leftarrow \text{assume } \vec{v} \neq \vec{0}$$

the projection of $\mathbf{u}$ onto $\mathbf{v}$.

1. Show that $\|\mathrm{Proj}_{\mathbf{v}}\mathbf{u}\| \leq \|\mathbf{u}\|$.
2. Under what circumstance(s) does the equality hold in Part (1)?



b) Using the formula of projection in **Q4**, show by direct calculation that $(\mathbf{u} - \mathrm{Proj}_{\mathbf{v}}\,\mathbf{u})\cdot\mathrm{Proj}_{\mathbf{v}}\,\mathbf{u} = 0$.

$$\mathrm{Proj}_{\vec{v}}\vec{u} = \|\vec{u}\|\cos\theta \; \frac{\vec{v}}{\|\vec{v}\|} = \left(\|\vec{u}\|\cdot\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}\right)\frac{\vec{v}}{\|\vec{v}\|} = \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v}.$$

Ans: 1) $\|\mathrm{Proj}_{\vec{v}}\vec{u}\| = \left\|\frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v}\right\| = \frac{|\vec{u}\cdot\vec{v}|}{\|\vec{v}\|^2}\|\vec{v}\| = \frac{|\vec{u}\cdot\vec{v}|}{\|\vec{v}\|}$

By Cauchy-Schwarz inequality,

$$|\vec{u}\cdot\vec{v}| \leq \|\vec{u}\|\,\|\vec{v}\|.$$

Hence $\|\mathrm{Proj}_{\vec{v}}\vec{u}\| \leq \frac{\|\vec{u}\|\,\|\vec{v}\|}{\|\vec{v}\|} = \|\vec{u}\|$.

2) "=" holds in 1)

$\iff$ "=" holds in Cauchy-Schwarz inequality

$\iff$ $\vec{u} = r\vec{v}$ or $\vec{v} = r\vec{u}$ for some $r \in \mathbb{R}$

i.e. $\vec{u} \,/\!/\, \vec{v}$.

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Ans: $(u - \mathrm{Proj}_v u)\cdot \mathrm{Proj}_v u$

$= \left(u - \frac{u\cdot v}{\|v\|^2}v\right)\cdot\left(\frac{u\cdot v}{\|v\|^2}v\right)$

$= u\cdot\left(\frac{u\cdot v}{\|v\|^2}v\right) - \left(\frac{u\cdot v}{\|v\|^2}v\right)\cdot\left(\frac{u\cdot v}{\|v\|^2}v\right)$ scalar

$= \frac{u\cdot v}{\|v\|^2}(u\cdot v) - \left(\frac{u\cdot v}{\|v\|^2}\right)^2(v\cdot v)$

$= \frac{(u\cdot v)^2}{\|v\|^2} - \frac{(u\cdot v)^2}{\|v\|^2}$

$= 0$, i.e. they are $\perp$

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