Mixed Partial Derivatives

In Exercises 51–54, verify that $w_{xy} = w_{yx}$.

51. $w = \ln(2x + 3y)$

Solution:

$$\frac{\partial w}{\partial x} = \frac{2}{2x+3y}, \quad \frac{\partial w}{\partial y} = \frac{3}{2x+3y}, \quad \frac{\partial^2 w}{\partial y \partial x} = \frac{-6}{(2x+3y)^2}, \text{ and } \frac{\partial^2 w}{\partial x \partial y} = \frac{-6}{(2x+3y)^2}$$

Using the Partial Derivative Definition

In Exercises 57–60, use the limit definition of partial derivative to compute the partial derivatives of the functions at the specified points.

62. Let $f(x, y) = x^2 + y^3$. Find the slope of the line tangent to this surface at the point $(-1, 1)$ and lying in the **a.** plane $x = -1$ **b.** plane $y = 1$.

Solution:

(a) In the plane $x = -1 \Rightarrow f_y(x, y) = 3y^2 \Rightarrow f_y(-1, 1) = 3(1)^2 = 3 \Rightarrow m = 3$

(b) In the plane $y = 1 \Rightarrow f_x(x, y) = 2x \Rightarrow f_y(-1, 1) = 2(-1) = -2 \Rightarrow m = -2$

63. Three variables Let $w = f(x, y, z)$ be a function of three independent variables and write the formal definition of the partial derivative $\partial f / \partial z$ at (x_0, y_0, z_0). Use this definition to find $\partial f / \partial z$ at $(1, 2, 3)$ for $f(x, y, z) = x^2yz^2$.

Solution:

$$f_z(x_0, y_0, z_0) = \lim_{h \to 0} \frac{f(x_0, y_0, z_0+h) - f(x_0, y_0, z_0)}{h};$$

$$f_z(1, 2, 3) = \lim_{h \to 0} \frac{f(1, 2, 3+h) - f(1, 2, 3)}{h} = \lim_{h \to 0} \frac{2(3+h)^2 - 2(9)}{h} = \lim_{h \to 0} \frac{12h + 2h^2}{h} = \lim_{h \to 0} (12 + 2h) = 12$$

Differentiating Implicitly

66. Find the value of $\partial x/\partial z$ at the point $(1, -1, -3)$ if the equation

$$xz + y \ln x - x^2 + 4 = 0$$

defines x as a function of the two independent variables y and z and the partial derivative exists.

Solution:

$$\left(\frac{\partial x}{\partial z}\right)z + x + \left(\frac{y}{x}\right)\frac{\partial x}{\partial z} - 2x\frac{\partial x}{\partial z} = 0 \Rightarrow \left(z + \frac{y}{x} - 2x\right)\frac{\partial x}{\partial z} = -x \Rightarrow \text{at } (1,-1,-3) \text{ we have } (-3-1-2)\frac{\partial x}{\partial z} = -1 \text{ or } \frac{\partial x}{\partial z} = \frac{1}{6}$$

Show that each function in Exercises 73–80 satisfies a Laplace equation.

Hint: Laplace equation in rectangular coordinates given by

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0.$$

Solution:

$$\frac{\partial f}{\partial x} = -\frac{1}{2}\left(x^2+y^2+z^2\right)^{-3/2}(2x) = -x\left(x^2+y^2+z^2\right)^{-3/2},$$

$$\frac{\partial f}{\partial y} = -\frac{1}{2}\left(x^2+y^2+z^2\right)^{-3/2}(2y) = -y\left(x^2+y^2+z^2\right)^{-3/2},$$

$$\frac{\partial f}{\partial z} = -\frac{1}{2}\left(x^2+y^2+z^2\right)^{-3/2}(2z) = -z\left(x^2+y^2+z^2\right)^{-3/2};$$

$$\frac{\partial^2 f}{\partial x^2} = -\left(x^2+y^2+z^2\right)^{-3/2} + 3x^2\left(x^2+y^2+z^2\right)^{-5/2},$$

$$\frac{\partial^2 f}{\partial y^2} = -\left(x^2+y^2+z^2\right)^{-3/2} + 3y^2\left(x^2+y^2+z^2\right)^{-5/2},$$

$$\frac{\partial^2 f}{\partial z^2} = -\left(x^2+y^2+z^2\right)^{-3/2} + 3z^2\left(x^2+y^2+z^2\right)^{-5/2}$$

$$\Rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \left[-\left(x^2+y^2+z^2\right)^{-3/2} + 3x^2\left(x^2+y^2+z^2\right)^{-5/2}\right]$$

$$+\left[-\left(x^2+y^2+z^2\right)^{-3/2} + 3y^2\left(x^2+y^2+z^2\right)^{-5/2}\right] + \left[-\left(x^2+y^2+z^2\right)^{-3/2} + 3z^2\left(x^2+y^2+z^2\right)^{-5/2}\right]$$

$$= -3\left(x^2+y^2+z^2\right)^{-3/2} + \left(3x^2+3y^2+3z^2\right)\left(x^2+y^2+z^2\right)^{-5/2} = 0$$