Tutorial 3

1. Present the coordinate surfaces in spherical coordinates:

$$\begin{cases} x = \rho \sin\phi \cos\theta \\ y = \rho \sin\phi \sin\theta \\ z = \rho \cos\phi \end{cases}$$

(i) $\rho =$ const.

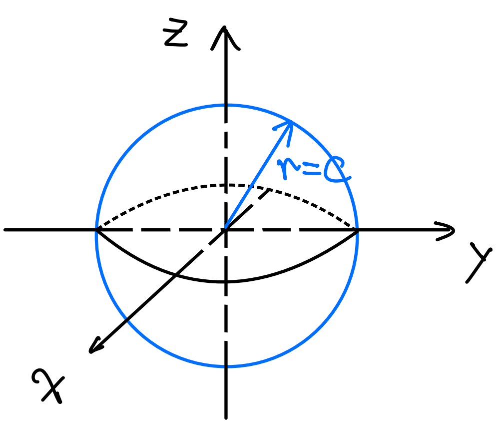
(ii) $\theta =$ const.

(iii) $\phi =$ const

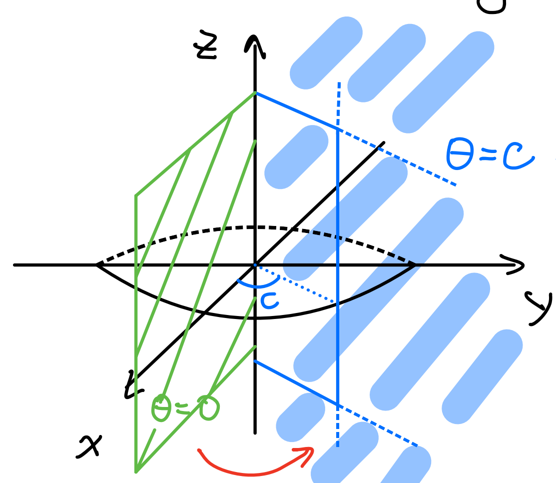
$\rho \in [0, +\infty)$, $\phi \in [0, \pi]$, $\theta \in [0, 2\pi)$

Sol

(i) $\rho = C$: sphere with radius C, centered at origin



(i)

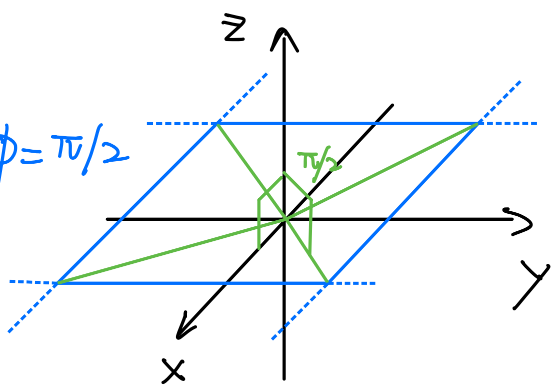


(ii)

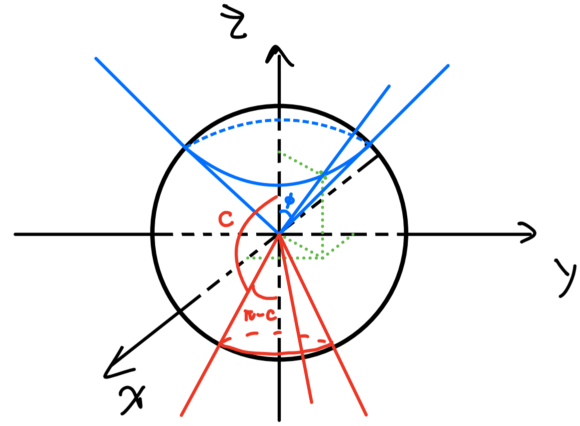
(ii) $\theta = C$: half plane, obtained by rotating the $\{\theta = 0\}$ half plane, which is $\{x \geq 0, y = 0, z \in \mathbb{R}\}$, counterclockwisely around the z-axis by angle C

(iii) $\phi = c$:

① $c = \frac{\pi}{2}$, plane $z = 0$



(iii) ①



(iii) ②

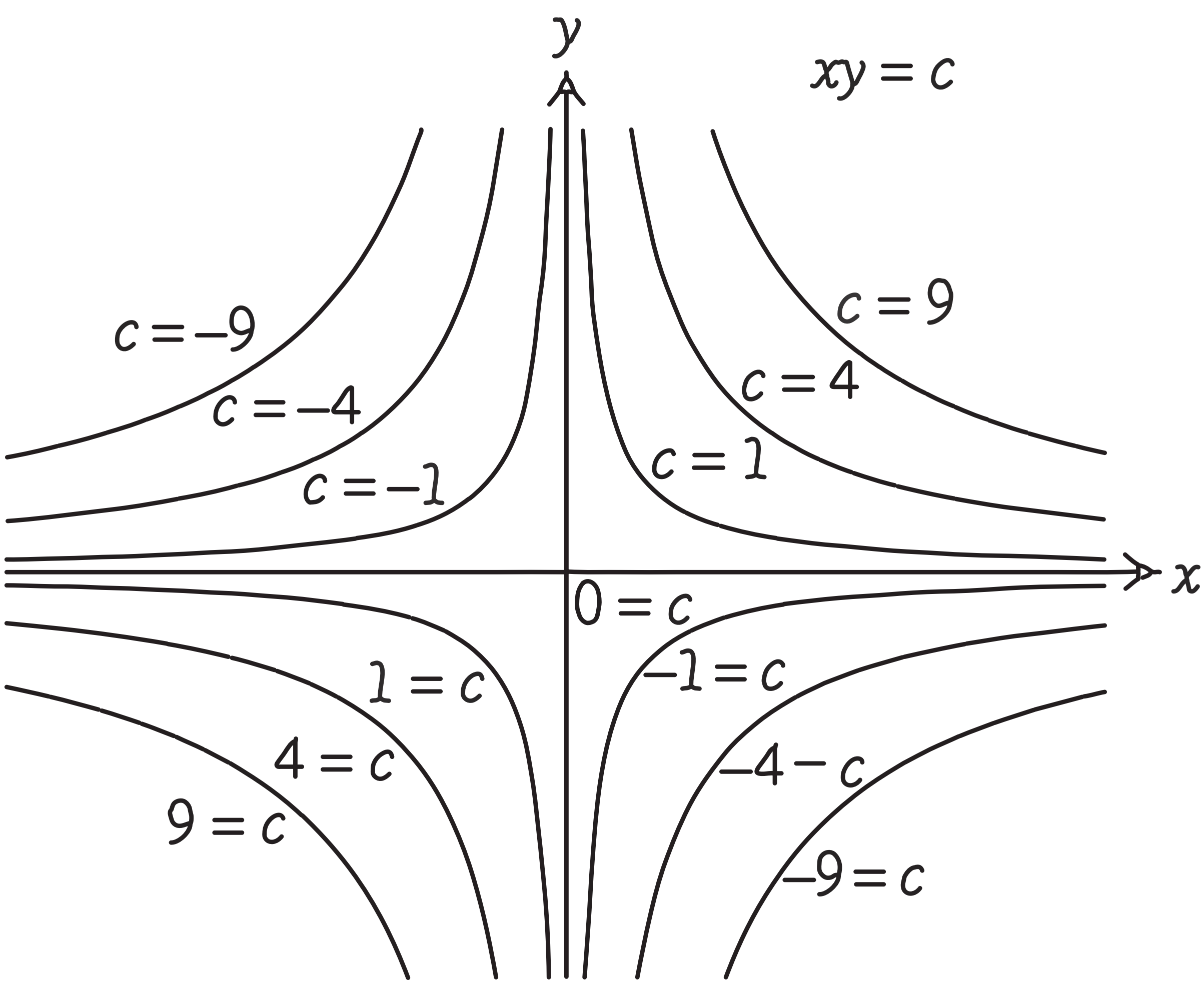
② $c \neq \pi/2$, baseless cone with vertex at the origin, z axis symmetric, and with angle c (when opening up top V, $c < \pi/2$) or angle $\pi - c$ (when opening down below, $c > \pi/2$)

$\square$

In Exercises 13–16, find and sketch the level curves $f(x, y) = c$ on the same set of coordinate axes for the given values of c. We refer to these level curves as a contour map.

15. $f(x, y) = xy, \quad c = -9, -4, -1, 0, 1, 4, 9$

<u>Sol</u> $xy = c$ hyperbola two branches



In Exercises 61–64, find an equation for the level surface of the function through the given point.

62. $f(x, y, z) = \ln(x^2 + y + z^2), \quad (-1, 2, 1)$

Sol. Level surface

$$c = f(x, y, z) = \ln(x^2 + y + z^2)$$

Substitute $(-1, 2, 1)$ into the above eq.

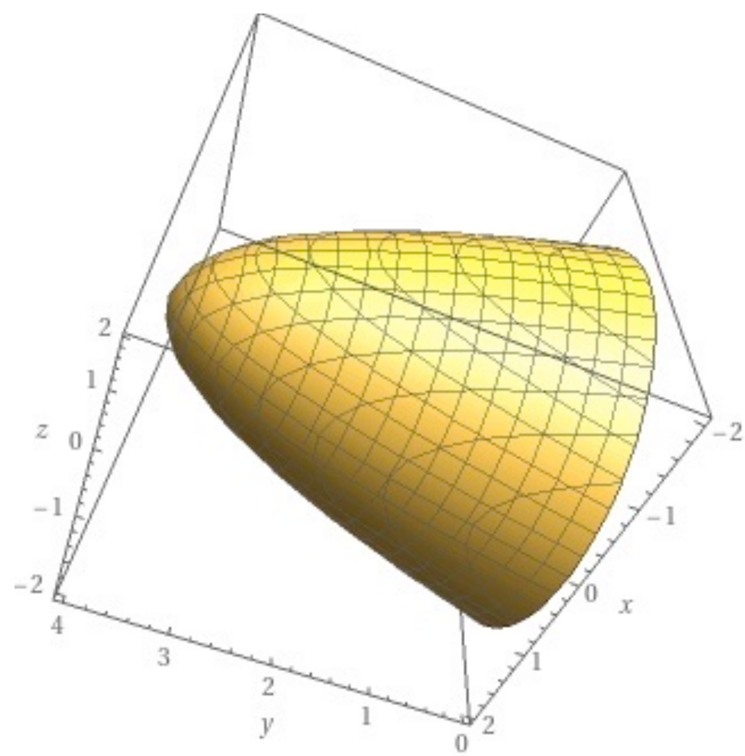
$$c = \ln((-1)^2 + 2 + 1^2) = \ln 4$$

$\Rightarrow$

$$\ln(x^2 + y + z^2) = \ln 4$$

Take exponential on both sides

$$x^2 + y + z^2 = 4 \qquad \text{paraboloid} \qquad \square$$



Find the limits in Exercises 13–24 by rewriting the fractions first.

17. $$\lim_{\substack{(x,y)\to(0,0)\\ x\neq y}} \frac{x-y+2\sqrt{x}-2\sqrt{y}}{\sqrt{x}-\sqrt{y}}$$

<u>Sol</u>

$$\frac{x-y+2\sqrt{x}-2\sqrt{y}}{\sqrt{x}-\sqrt{y}}$$

$$= \frac{x-y}{\sqrt{x}-\sqrt{y}} + 2 = \frac{(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y})}{\sqrt{x}-\sqrt{y}} + 2$$

$$= \sqrt{x}+\sqrt{y}+2$$

so

$$\lim_{\substack{(x,y)\to(0,0)\\ x\neq y}} \frac{x-y+2\sqrt{x}-2\sqrt{y}}{\sqrt{x}-\sqrt{y}}$$

$$= \lim_{\substack{(x,y)\to(0,0)\\ x\neq y}} \sqrt{x}+\sqrt{y}+2 = 2$$

$\square$

By considering different paths of approach, show that the functions in Exercises 41–48 have no limit as $(x, y) \to (0, 0)$.

41. $f(x, y) = -\dfrac{x}{\sqrt{x^2 + y^2}}$

Pf [Observation $\sqrt{x^2+y^2} \approx O(|x|)$, $x \approx O(|x|)$]

Just take $y = \alpha x$ $\alpha \neq 0$

$$-\frac{x}{\sqrt{x^2+\alpha^2 x^2}} = -\frac{x}{\sqrt{1+\alpha^2}\,|x|}$$

consider the paths along $y = \alpha x$ $x > 0$ and $x < 0$

$$\lim_{\substack{(x,y)\to(0,0) \\ y=\alpha x \\ x>0}} f(x,y) = \lim_{\substack{(x,y)\to(0,0) \\ y=\alpha x \\ x>0}} -\frac{x}{\sqrt{1+\alpha^2}\,|x|} = -\frac{1}{\sqrt{1+\alpha^2}}$$

$$\neq$$

$$\lim_{\substack{(x,y)\to(0,0) \\ y=\alpha x \\ x<0}} f(x,y) = \lim_{\substack{(x,y)\to(0,0) \\ y=\alpha x \\ x<0}} -\frac{x}{\sqrt{1+\alpha^2}\,|x|} = +\frac{1}{\sqrt{1+\alpha^2}}$$

$\Rightarrow$ $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist

$\square$

51. Let $f(x, y) = \begin{cases} 1, & y \geq x^4 \\ 1, & y \leq 0 \\ 0, & \text{otherwise.} \end{cases}$

Find each of the following limits, or explain that the limit does not exist.

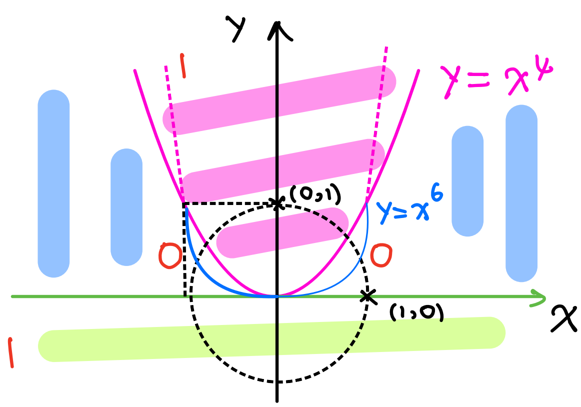
a. $\displaystyle\lim_{(x, y)\to(0,1)} f(x, y)$

b. $\displaystyle\lim_{(x, y)\to(2,3)} f(x, y)$

c. $\displaystyle\lim_{(x, y)\to(0,0)} f(x, y)$

Sol $(0,1)$ has neighborhoods inside $\{y \geq x^4\}$;

$(2,3)$ inside $\mathbb{R}^2 \setminus (\{y \geq x^4\} \cup \{y \leq 0\})$; $(0,0)$ is tricky,



it is the intersection of $\{y = x^4\}$ and $\{y = 0\}$ the two constitute the discontinuous set of function f.

a. $f \equiv 1$ on $\{y \geq x^4\} \Rightarrow \displaystyle\lim_{(x,y)\to(0,1)} f(x,y) = 1$

b. $f \equiv 0$ on $\mathbb{R}^2 \setminus (\{y \geq x^4\} \cup \{y \leq 0\}) \Rightarrow \displaystyle\lim_{(x,y)\to(2,3)} f = 0$

c. Consider the path $y = x^4$

$$f(x,y)=1 \quad \text{on} \quad y=x^4 \Rightarrow \lim_{\substack{(x,y)\to(0,0)\\ y=x^4}} f(x,y)=1$$

Now one can take another path inside $\mathbb{R}^2 \setminus (\{y \geq x^4\} \cup \{y \leq 0\})$ (the blue one), for example $y=x^6$, $(x,y)\in B_1(0)$, because $x^6 < x^4$ for $|x|<1$, then $f(x,y)=0$ on $y=x^6$, $(x,y)\in B_1(0)$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=x^6,\, |x|<1}} f(x,y)=0$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} f \text{ does not exist.}$$

$\square$