HW 2 §12.5 Q33, 39, 57 §13.1 Q19 §13.3 Q2, 18

§11.3 Q27, 59 §11.4 Q6, 19

1. Find the distance from the point to the line

$(2,1,3)$ $\quad x = 2+2t, \; y = 1+6t, \; z = 3$

<u>Sol</u> The parametric form of the line is

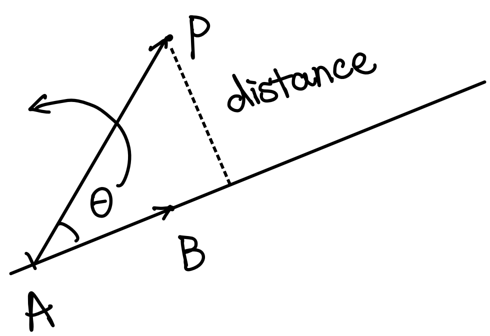
$$(x, y, z) = (2, 1, 3) + t(2, 6, 0)$$

So point $(2,1,3)$ is on the line, and the distance is 0. $\square$

In general,

<u>Method 1</u>

$\cos\theta = \dfrac{u \cdot v}{|u||v|}$



$$\begin{aligned} d &:= |\overrightarrow{AP}| \cdot \sin\theta \qquad \theta \in [0, \pi] \\ &= |\overrightarrow{AP}| \sqrt{1 - \cos^2\theta} \\ &= |\overrightarrow{AP}| \sqrt{1 - \left(\frac{\overrightarrow{AP} \cdot \overrightarrow{AB}}{|\overrightarrow{AP}| \, |\overrightarrow{AB}|} \right)^2} \end{aligned}$$

<u>Method 2</u>

$$|\overrightarrow{AP} \times \overrightarrow{AB}| = |\overrightarrow{AP}| \, |\overrightarrow{AB}| \sin\theta$$

$\Rightarrow$

$$|\overrightarrow{AP}| \cdot \sin\theta = \frac{|\overrightarrow{AP} \times \overrightarrow{AB}|}{|\overrightarrow{AB}|}$$

but you have to compute $\overrightarrow{AP} \times \overrightarrow{AB}$ by $\begin{vmatrix} i & j & k \\ & \overrightarrow{AP} & \\ & \overrightarrow{AB} & \end{vmatrix}$

2. Find parametric equations for the line that is tangent to the given curve at the given parameter value $t = t_0$

$$\vec{r}(t) = \ln t \, \vec{i} + \frac{t-1}{t+2} \vec{j} + t \ln t \, \vec{k}, \qquad t_0 = 1$$

Sol Find a point on the line and the direction of the line

$$P : \vec{r}(t_0) = \ln(1) \, \vec{i} + \frac{1-1}{1+3} \vec{j} + 1 \cdot \ln(1) \, \vec{k} = (0, 0, 0) \text{ --- } (1)$$

Tangent vector (velocity vector) is given by

$$\frac{d}{dt} \vec{r}(t) = \dot{\vec{r}}(t) = \vec{r}\,'(t)$$

$$= \frac{1}{t} \cdot \vec{i} + \frac{3}{(t+2)^2} \vec{j} + (\ln t + 1) \, \vec{k}$$

$$\vec{r}\,'(t_0) = \vec{i} + \frac{1}{3} \vec{j} + \vec{k} \text{ --- } (2)$$

Consequently

$$\ell(s) = P + s \cdot \vec{r}\,'(t_0)$$

$$= (0, 0, 0) + s \cdot \left(1, \frac{1}{3}, 1\right)$$

$$\ell : \quad x = s, \quad y = \frac{1}{3} s, \quad z = s$$

$\square$

3. Find the arc length parameter along the curve from the point where $t=0$ by evaluating the integral

$$s = \int_0^t |\vec{v}(\tau)|\, d\tau . \qquad (\vec{v} \text{ the velocity vector})$$

Then find the length of the indicated portion of the curve

$$\vec{r}(t) = (4\cos t)\vec{i} + (4\sin t)\vec{j} + 3t\vec{k}, \qquad 0 \le t \le \pi/2$$

Remark on arc length parametrization

$$\frac{d\vec{r}}{ds} = \frac{d\vec{r}}{dt}\frac{dt}{ds} = \frac{d\vec{r}}{dt}\frac{1}{\frac{ds}{dt}} = \frac{d\vec{r}/dt}{|\vec{v}(t)|} = \frac{\vec{v}}{|\vec{v}|}$$

is the unit tangent vector

$$s = \int_0^s \left|\frac{d\vec{r}}{ds}\right| d\tau = \int_0^s 1\, d\tau = s$$

Sol $\quad \vec{v}(t) = \dfrac{d\vec{r}}{dt} = (-4\sin t,\ 4\cos t,\ 3)$

$$s = \int_0^t |\vec{v}(\tau)|\, d\tau = \int_0^t \sqrt{(-4\sin\tau)^2 + (4\cos\tau)^2 + 3^2}\, d\tau = 5t$$

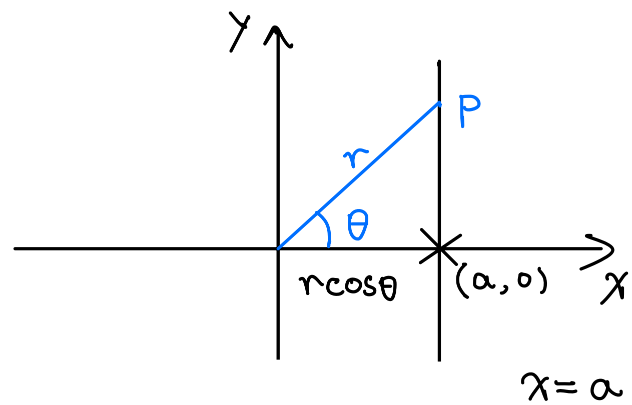
$$\text{Length} = s(t=\pi/2) = \frac{5}{2}\pi$$

□

4. **Vertical and horizontal lines**

a. Show that every vertical line in the xy-plane has a polar equation of the form $r = a \sec\theta$.

b. Find the analogous polar equation for horizontal lines in the xy-plane.



Sol.

a. Pf All vertical lines have the form $x=a$ for some $a \in \mathbb{R}$

Polar coordinate $x = r\cos\theta$.

Then polar equation is $a = x = r\cos\theta$

$$\Rightarrow \quad r = \frac{a}{\cos\theta} = a\sec\theta$$

b. Sol All horizontal lines have the form $y = b$

Polar coordinate $y = r\sin\theta$

$$\Rightarrow \quad b = y = r\sin\theta$$

$$\Rightarrow \quad r = \frac{b}{\sin\theta} = b\csc\theta$$

5. Find the slopes of the curves at the given points. Sketch the curves along with their tangents at these points

$r = \cos 2\theta$ at $\theta = 0, \pm\frac{\pi}{2}, \pi$

Slope $\quad \vec{r}(t) = (x(t), y(t)) \dashrightarrow y = f(x)$ graph

$$\text{slope} = \frac{dy}{dx} = f'(x)$$

In polar coordinates, polar equation $r = r(\theta)$

$$\Rightarrow \begin{cases} x = r\cos\theta = r(\theta)\cos\theta =: x(\theta) \\ y = r\sin\theta = r(\theta)\sin\theta =: y(\theta) \end{cases}$$

parametrized by θ.

$$\text{slope} = \frac{dy}{dx} = \frac{dy}{d\theta}\frac{d\theta}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

In the textbook, negative r is allowed

$$(x, y) = (r\cos\theta, r\sin\theta) \qquad r < 0$$

$$= (|r|\cos(\theta+\pi), |r|\sin(\theta+\pi))$$

For $r < 0$, $(r, \theta) \Leftrightarrow (|r|, \theta + \pi)$

5. Find the slopes of the curves at the given points. Sketch the curves along with their tangents at these points

$r = \cos 2\theta$ at $\theta = 0, \pm\frac{\pi}{2}, \pi$ Four-leaved rose

<u>Sol</u>

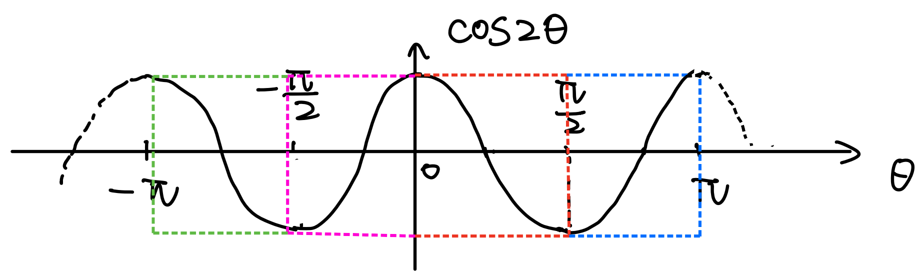
$$\text{slope} = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

$$\begin{cases} x = r\cos\theta = \cos 2\theta \cos\theta \\ y = r\sin\theta = \cos 2\theta \sin\theta \end{cases} \quad \begin{bmatrix} = \frac{1}{2}(\cos 3\theta + \cos\theta) \\ = \frac{1}{2}(\sin 3\theta - \sin\theta) \end{bmatrix}$$

$$\Rightarrow \text{slope} = \frac{-\sin 2\theta \cdot 2\sin\theta + \cos 2\theta \cos\theta}{-\sin 2\theta \cdot 2\cos\theta - \cos 2\theta \sin\theta} = \frac{6\cos^3\theta - 5\cos\theta}{6\sin^3\theta - 5\sin\theta}$$

Domain of theta

$\theta \in [-\pi, \pi]$,



then

$\text{slope}(\theta = 0) = \infty$ $\qquad$ $\text{slope}(\theta = \pi) = \infty$

$\text{slope}(\theta = -\frac{\pi}{2}) = 0$ $\qquad$ $\text{slope}(\theta = \frac{\pi}{2}) = 0$

Also $\text{slope}(\theta = -\frac{3}{4}\pi) = 1 = \text{slope}(\theta = \frac{\pi}{4})$

$\text{slope}(\theta = -\frac{\pi}{4}) = -1 = \text{slope}(\theta = \frac{3}{4}\pi)$

Sketch

Rotation ~~Symmetry~~ ☆ $\Rightarrow$

$\boxed{\begin{array}{l}\text{Consider } \theta \in [0, \frac{\pi}{2}] \\ \text{Take special points} \\ 2\theta \to \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{5\pi}{6}, \pi\end{array}}$

$\theta = -\pi \qquad r = 1$

$\vdots$

$\theta = 0 \qquad r = 1$

$\theta = \frac{\pi}{12} \qquad r = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

$\theta = \frac{\pi}{6} \qquad r = \cos\frac{\pi}{3} = \frac{1}{2}$

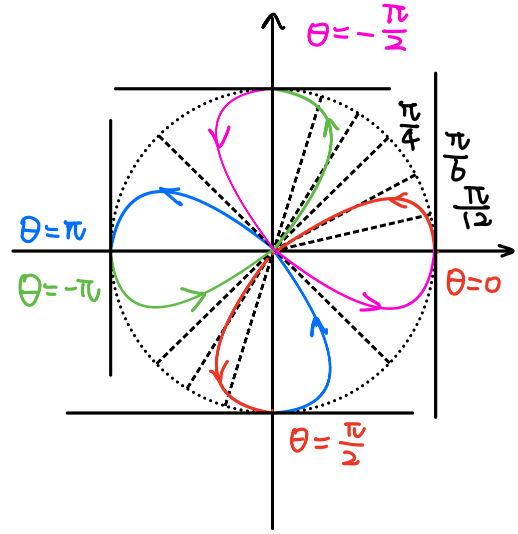
$\theta = \frac{\pi}{4} \qquad r = \cos\frac{\pi}{2} = 0$

$\theta = \frac{\pi}{3} \qquad r = \cos\frac{2\pi}{3} = -\frac{1}{2}$

$\theta = \frac{5\pi}{12} \qquad r = -\frac{\sqrt{3}}{2}$

$\theta = \frac{\pi}{2} \qquad r = -1$

$\vdots$



color correspondence