HW1 Remainder §12.3 Q 6, 13, 28 §12.4 Q 2, 11, 21

§12.5 Q 3, 16, 22, 24

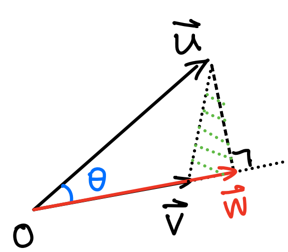
1. Given $\vec{v}, \vec{u} \in \mathbb{R}^n$, what is the projection of $\vec{u}$ onto $\vec{v}$?

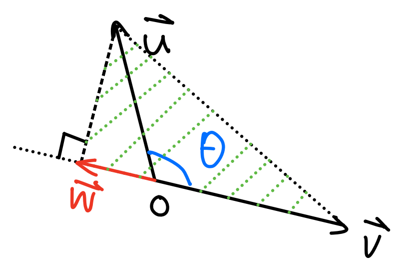
<u>Def</u> The vector projection of $\vec{u}$ onto a nonzero vector $\vec{v}$, denoted by $\mathrm{proj}_{\vec{v}}\vec{u}$, is given by the following formula:

$$\mathrm{proj}_{\vec{v}}\vec{u} := \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\vec{v}$$

Explanation: $\mathrm{proj}_{\vec{v}}\vec{u}$ is a vector $\vec{w}$ that has direction $\vec{v}$, and "length" $|\vec{u}|\cos\theta$

Here θ is the angle between $\vec{u}$ and $\vec{v}$





Pythagoras thm in the Green Region

Recall the proof of $\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{|\vec{u}|\,|\vec{v}|}$

This length $|\vec{u}|\cos\theta$ could be negative ("oriented length")

and is called the scalar component of $\vec{u}$ in the direction of $\vec{v}$

or .... $\vec{u}$ onto $\vec{v}$

☆ Notations $|\vec{u}| \equiv$ length of $\vec{u} \equiv \|\vec{u}\|$

Consequently

$$\text{proj}_{\vec{v}}\,\vec{u} = \vec{w} = |\vec{u}|\cos\theta\,\frac{\vec{v}}{|\vec{v}|} = |\vec{u}|\,\frac{\vec{u}\cdot\vec{v}}{|\vec{u}|\,|\vec{v}|}\,\frac{\vec{v}}{|\vec{v}|} = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\,\vec{v}$$

EXAMPLE 5 Find the vector projection of $\mathbf{u} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ onto $\mathbf{v} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and the scalar component of $\mathbf{u}$ in the direction of $\mathbf{v}$.

Solution We find $\text{proj}_{\mathbf{v}}\,\mathbf{u}$ from Equation (1):

$$\begin{aligned}\text{proj}_{\mathbf{v}}\,\mathbf{u} &= \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = \frac{6-6-4}{1+4+4}(\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}) \\ &= -\frac{4}{9}(\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}) = -\frac{4}{9}\mathbf{i} + \frac{8}{9}\mathbf{j} + \frac{8}{9}\mathbf{k}.\end{aligned}$$

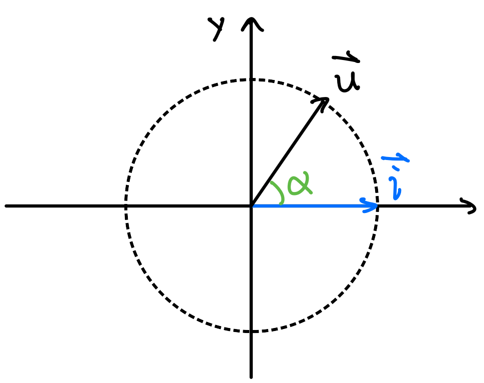
We find the scalar component of $\mathbf{u}$ in the direction of $\mathbf{v}$ from Equation (2):

$$\begin{aligned}|\mathbf{u}|\cos\theta &= \mathbf{u}\cdot\frac{\mathbf{v}}{|\mathbf{v}|} = (6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k})\cdot\left(\frac{1}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}\right) \\ &= 2 - 2 - \frac{4}{3} = -\frac{4}{3}.\end{aligned}$$

■

2. **Unit vectors in the plane** Show that a unit vector in the plane can be expressed as $\mathbf{u} = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}$, obtained by rotating $\mathbf{i}$ through an angle θ in the counterclockwise direction. Explain why this form gives *every* unit vector in the plane.

Pf Given any unit vector $\vec{u} \in \mathbb{R}^2$, $\vec{u} = (u^1, u^2) \equiv u^1\vec{i} + u^2\vec{j}$ and $|\vec{u}| = 1$, to determine θ, just consider the angle α between $\vec{u}$ and $\vec{i}$,



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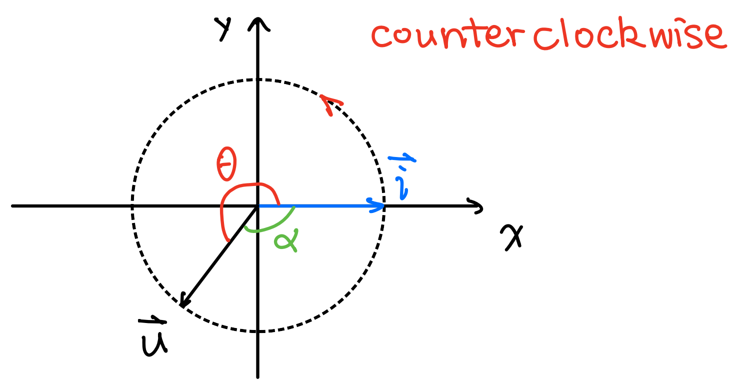


II

then $\cos\alpha = \dfrac{\vec{u}\cdot\vec{i}}{|\vec{u}||\vec{i}|} = u^1$, $\alpha \in [0, \pi]$ —— (2.1)

Note that $|\vec{u}| = 1 \Leftrightarrow (u^1)^2 + (u^2)^2 = 1$

By (2.1) $\Rightarrow (u^2)^2 = 1 - \cos^2\alpha = \sin^2\alpha \Rightarrow |u^2| = \sin\alpha \geq 0$

Case I $u^2 \geq 0$ $\Rightarrow u^2 = |u^2| = \sin\alpha$.

$$\vec{u} = u^1\vec{i} + u^2\vec{j} = \cos\alpha\,\vec{i} + \sin\alpha\,\vec{j}$$

Simply take $\theta = \alpha$.

__Case II__ $\underline{u^2 < 0}$ $\Rightarrow$ $u^2 = -|u^2| = -\sin\alpha$

$$\vec{u} = \cos\alpha\,\vec{i} - \sin\alpha\,\vec{j}$$

$$= \cos(-\alpha)\,\vec{i} + \sin(-\alpha)\,\vec{j}$$

$$= \cos(-\alpha+2\pi)\,\vec{i} + \sin(-\alpha+2\pi)\,\vec{j}$$

Take $\theta = -\alpha + 2\pi$

Since $\vec{u}$ is arbitrarily given, we have the following:

__Conclusion__ $\alpha :=$ angle $(\vec{u}, \vec{i}) = u^1$, then any unit vector $\vec{u}$ in the plane takes the form $\vec{u} = \cos\theta\,\vec{i} + \sin\theta\,\vec{j}$

where

$$\theta = \begin{cases} \alpha & \text{if } u^2 \geq 0 \quad (\text{otherwise}) \\ -\alpha + 2\pi & \text{otherwise} \quad (\vec{u} \text{ in the lower half space}) \end{cases}$$

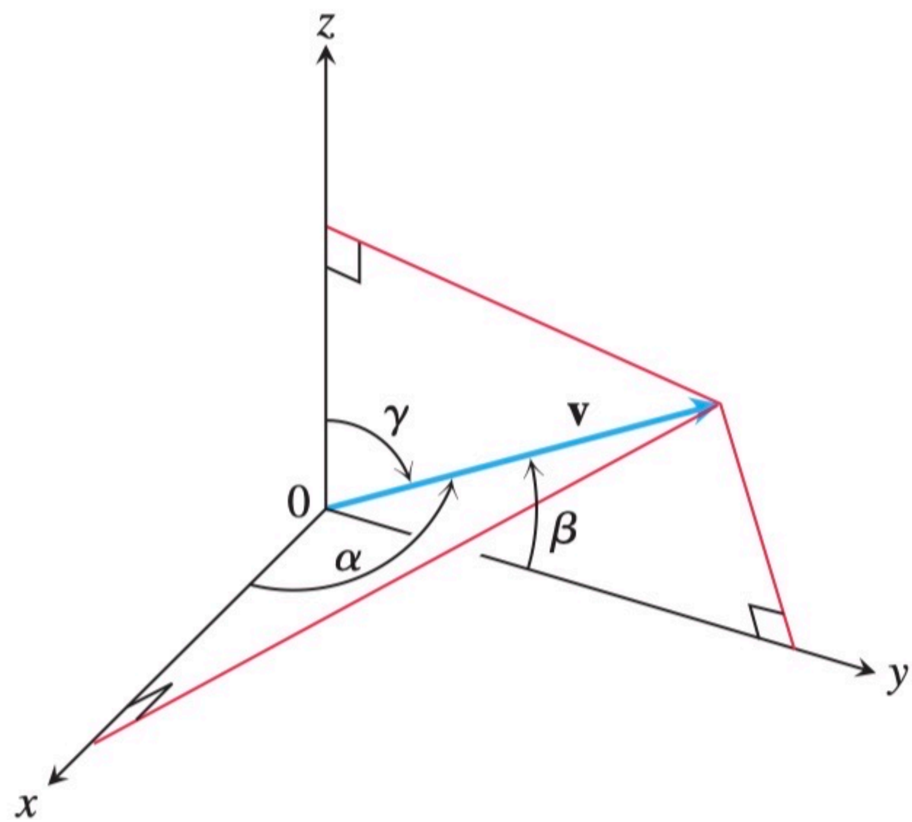
$\square$

3. **Direction angles and direction cosines** The *direction angles* α, β, and γ of a vector $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ are defined as follows:

α is the angle between $\mathbf{v}$ and the positive x-axis $(0 \le \alpha \le \pi)$

β is the angle between $\mathbf{v}$ and the positive y-axis $(0 \le \beta \le \pi)$

γ is the angle between $\mathbf{v}$ and the positive z-axis $(0 \le \gamma \le \pi)$.



a. Show that

$$\cos\alpha = \frac{a}{|\mathbf{v}|}, \qquad \cos\beta = \frac{b}{|\mathbf{v}|}, \qquad \cos\gamma = \frac{c}{|\mathbf{v}|},$$

and $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$. These cosines are called the *direction cosines* of $\mathbf{v}$.

b. Unit vectors are built from direction cosines Show that if $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is a unit vector, then a, b, and c are the direction cosines of $\mathbf{v}$.

Pf a. Just compute the angles between $\vec{v}$ and $\vec{i}, \vec{j}, \vec{k}$

$$\cos\alpha = \frac{\vec{v}\cdot\vec{i}}{|\vec{v}||\vec{i}|} = \frac{a}{|\vec{v}|}, \quad \text{.......................................}\ ,$$

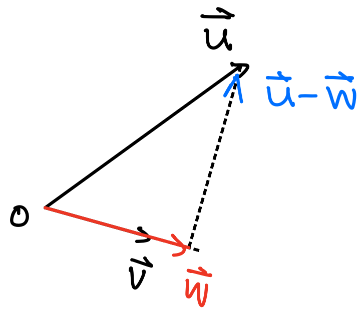
$$\begin{aligned}
\cos^2\alpha + \cos^2\beta + \cos^2\gamma &= \left(\frac{a}{|\vec{v}|}\right)^2 + \left(\frac{b}{|\vec{v}|}\right)^2 + \left(\frac{c}{|\vec{v}|}\right)^2 \\
&= \frac{\overbrace{a^2+b^2+c^2}^{\vec{v}\cdot\vec{v}}}{|\vec{v}|^2} \\
&= 1
\end{aligned}$$

b. From part a and the condition $|\vec{v}| = 1$,

$$\cos\alpha = \frac{a}{|\vec{v}|} = a, \quad \cos\beta = b, \quad \cos\gamma = c$$

$\square$

4. Using the definition of the projection of **u** onto **v**, show by direct calculation that $(\mathbf{u} - \text{proj}_{\mathbf{v}}\,\mathbf{u}) \cdot \text{proj}_{\mathbf{v}}\,\mathbf{u} = 0$.



$$\vec{u} - \text{proj}_{\vec{v}}\,\vec{u} \perp \text{proj}_{\vec{v}}\,\vec{u}$$

Pf Recall the definition

$$\text{proj}_{\vec{v}}\,\vec{u} := \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\,\vec{v}$$

$$(\vec{u} - \text{proj}_{\vec{v}}\,\vec{u}) \cdot \text{proj}_{\vec{v}}\,\vec{u} = \vec{u} \cdot \text{proj}_{\vec{v}}\,\vec{u} - |\text{proj}_{\vec{v}}\,\vec{u}|^2$$

$$= \vec{u} \cdot \left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\,\vec{v} \right) - \left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2} \right)^2 |\vec{v}|^2$$

$$= \frac{(\vec{u}\cdot\vec{v})^2}{|\vec{v}|^2} - \frac{(\vec{u}\cdot\vec{v})^2}{|\vec{v}|^4}\,|\vec{v}|^2 = 0$$

$\square$