MATH2010 — 2023-24

Tutorial 9 — Term 1

Exercise 14.8

Lagrange Multipliers.

Suppose that $f, g : \Omega \to \mathbb{R}$ be C^1 functions, $\Omega \subset \mathbb{R}^n$ open, and $\nabla g \neq 0$ when $g(\vec{x}) = 0$.

- Finding local extrema of $f(\vec{x})$ with constraint $g(\vec{x}) = c$ at $\vec{x} = \vec{a}$.

Then: $\begin{cases} \bullet\ \vec{\nabla} f(\vec{a}) = \lambda \vec{\nabla} g(\vec{a}) & \text{for some } \lambda \in \mathbb{R} \\ \bullet\ g(\vec{a}) = c \end{cases}$

$$\Updownarrow$$

- Finding local extrema of $\boxed{F(\vec{x}, \lambda) = f(\vec{x}) - \lambda(g(\vec{x}) - c)}$

Then: $\vec{\nabla} F = \vec{0}$

i.e. $\begin{cases} \dfrac{\partial F}{\partial x_i} = 0 & \forall\ i = 1, \dots, n \\ \dfrac{\partial F}{\partial \lambda} = 0 \end{cases}$

Q6 Find the points on the curve $x^2 y = 2$ nearest the origin.

minimize $f(x,y) = x^2 + y^2$ with constraint $g(x,y) = x^2 y = 2$.

Consider $F(x,y,\lambda) = x^2 + y^2 - \lambda(x^2 y - 2)$

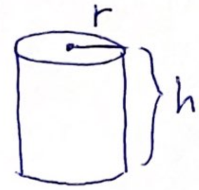
$$\begin{cases} \dfrac{\partial F}{\partial x} = 2x - 2\lambda x y = 0 \\ \dfrac{\partial F}{\partial y} = 2y - \lambda x^2 = 0 \\ \dfrac{\partial F}{\partial \lambda} = -(x^2 y - 2) = 0 \end{cases} \Rightarrow \begin{cases} 2x = 2\lambda x y \\ 2y = \lambda x^2 \end{cases}$$

$\Rightarrow \lambda = \dfrac{2y}{x^2}$ ($x \neq 0$, otherwise $x = 0, y = 0, g(0,0) = 0 \neq 2$)

$\Rightarrow 2x = 2\left(\dfrac{2y}{x^2}\right) x y \Rightarrow x^2 = 2y^2$

$\Rightarrow g(x,y) = (2y^2) \cdot y = 2y^3 = 2 \Rightarrow y = 1 \Rightarrow x = \pm\sqrt{2}$ $\therefore (\pm\sqrt{2}, 1)$

Q9 Find the dimensions of the closed right circular cylindrical can of smallest surface area whose volume is $16\pi\ cm^3$.



minimize $S = 2\pi r h + 2\pi r^2$

with constraint $V = \pi r^2 h = 16\pi$

$g = r^2 h = 16$

Consider $F(r,h,\lambda) = 2\pi r h + 2\pi r^2 - \lambda(r^2 h - 16)$

$$\begin{cases} \dfrac{\partial F}{\partial r} = 2\pi h + 4\pi r - 2\lambda r h = 0 \\ \dfrac{\partial F}{\partial h} = 2\pi r - \lambda r^2 = 0 \quad \Rightarrow \quad \lambda = \dfrac{2\pi}{r} \quad (r \neq 0) \\ \dfrac{\partial F}{\partial \lambda} = -(r^2 h - 16) = 0 \end{cases}$$

otherwise no cylindrical can

$$\Rightarrow \quad 2\pi h + 4\pi r = 2\left(\frac{2\pi}{r}\right) r h = 4\pi h$$

$$4\pi r = 2\pi h$$

$$2r = h$$

$$\Rightarrow \quad r^2(2r) = 16 \quad \Rightarrow \quad r^3 = 8 \quad \Rightarrow \quad r = 2 \quad \Rightarrow \quad h = 4$$

the only extremum : $S = 24\pi$

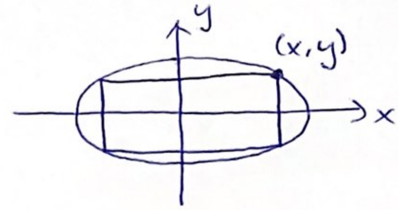
Take $r = 4$, $h = 1$ $\Rightarrow$ $S = 40\pi > 24\pi$

$\therefore$ $24\pi\ cm^3$ is the <u>minimum</u> surface area.

Q12 Find the dimensions of the rectangle of largest perimeter that can be inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with sides parallel to the coordinate axes. What is the largest perimeter?

maximize $P = 4x + 4y$

with constraint $g(x,y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



Consider $F(x,y,\lambda) = 4x + 4y - \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)$

$$\begin{cases} \frac{\partial F}{\partial x} = 4 - \frac{2\lambda x}{a^2} = 0 \\ \frac{\partial F}{\partial y} = 4 - \frac{2\lambda y}{b^2} = 0 \quad \Rightarrow \quad \lambda = \frac{2b^2}{y} \\ \frac{\partial F}{\partial \lambda} = -\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) = 0 \end{cases}$$

$$\Rightarrow \quad 4 = \frac{2\left(\frac{2b^2}{y}\right)x}{a^2} \quad \Rightarrow \quad y = \frac{b^2}{a^2}x$$

$$\Rightarrow \quad \frac{x^2}{a^2} + \frac{\left(\frac{b^2}{a^2}x\right)^2}{b^2} = 1$$

$$\frac{x^2}{a^2} + \frac{b^2x^2}{a^4} = 1$$

$$(a^2 + b^2)x^2 = a^4$$

$$x = \frac{a^2}{\sqrt{a^2+b^2}} \qquad (x > 0)$$

$$\Rightarrow \quad y = \frac{b^2}{\sqrt{a^2+b^2}}$$

$\therefore$ width $= 2x = \frac{2a^2}{\sqrt{a^2+b^2}}$,

height $= 2y = \frac{2b^2}{\sqrt{a^2+b^2}}$

largest perimeter $= 4x + 4y = \frac{4a^2 + 4b^2}{\sqrt{a^2+b^2}} = 4\sqrt{a^2+b^2}$

Exercise 14.9

Taylor's polynomial (2nd order) at $\vec{x}_0 = \vec{a}$

$$\boxed{P_2(\vec{x}) = f(\vec{a}) + \underbrace{Df(\vec{a})}_{\text{matrix of partial derivatives}} (\vec{x}-\vec{a}) + \frac{1}{2}(\vec{x}-\vec{a})^T \underbrace{Hf(\vec{a})}_{\text{Hessian matrix}} (\vec{x}-\vec{a})}$$

Q4 $f(x,y) = \sin x \cos y$, at the origin.

$$Df(x,y) = \begin{pmatrix} \cos x \cos y & -\sin x \sin y \end{pmatrix}$$

$$Df(0,0) = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

$$Hf(x,y) = \begin{pmatrix} -\sin x \cos y & -\cos x \sin y \\ -\cos x \sin y & -\sin x \cos y \end{pmatrix}$$

$$Hf(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$f(0,0) = 0$$

$$P_2(x,y) = 0 + \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + 0 = x$$

Q7 $f(x,y) = \sin(x^2+y^2)$, at the origin.

$$Df(x,y) = \begin{pmatrix} 2x\cos(x^2+y^2) & 2y\cos(x^2+y^2) \end{pmatrix}$$

$$Df(0,0) = \begin{pmatrix} 0 & 0 \end{pmatrix}$$

$$Hf(x,y) = \begin{pmatrix} 2\cos(x^2+y^2) - 4x^2\sin(x^2+y^2) & -4xy\sin(x^2+y^2) \\ -4xy\sin(x^2+y^2) & 2\cos(x^2+y^2) - 4y^2\sin(x^2+y^2) \end{pmatrix}$$

$$Hf(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$f(0,0) = 0$$

$$P_2(x,y) = 0 + \begin{pmatrix} 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{2}\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = x^2 + y^2$$