MATH 2010 2023-24

Tutorial 8 Term 1

Exercise 14.7

Find all the local maxima, local minima, and saddle points of the functions

Second Derivative Test for 2-variables

Let
- $f: \Omega \to \mathbb{R}$ be C^2, $\Omega \subseteq \mathbb{R}^2$, open
- $(a,b) \in \Omega$ such that $\vec{\nabla} f(a,b) = \vec{0}$

Then
(1) $f_{xx}f_{yy} - f_{xy}^2 > 0$ & $f_{xx} > 0$ at (a,b) $\Rightarrow$ local min
(2) $f_{xx}f_{yy} - f_{xy}^2 > 0$ & $f_{xx} < 0$ at (a,b) $\Rightarrow$ local max
(3) $f_{xx}f_{yy} - f_{xy}^2 < 0$ at (a,b) $\Rightarrow$ saddle point
(4) $f_{xx}f_{yy} - f_{xy}^2 = 0$ at (a,b) $\Rightarrow$ inconclusive.

Q3 $f(x,y) = x^2 + xy + 3x + 2y + 5$

$$\begin{cases} f_x = 2x + y + 3 = 0 \\ f_y = x + 2 = 0 \end{cases} \Rightarrow \begin{cases} x = -2 \\ y = 1 \end{cases}$$

$f_{xx} = 2$, $f_{yy} = 0$, $f_{xy} = 1$

$f_{xx}f_{yy} - f_{xy}^2 = 0 - 1 = -1 < 0$

$\Rightarrow (-2, 1)$ is a saddle point.

Q12 $f(x,y) = 1 - \sqrt[3]{x^2+y^2}$

$$\begin{cases} f_x = -\frac{1}{3}(x^2+y^2)^{-\frac{2}{3}}(2x) = -\dfrac{2x}{3(x^2+y^2)^{\frac{2}{3}}} = 0 \\ f_y = -\frac{1}{3}(x^2+y^2)^{-\frac{2}{3}}(2y) = -\dfrac{2y}{3(x^2+y^2)^{\frac{2}{3}}} = 0 \end{cases} \Rightarrow \text{No Solutions.}$$

Consider $\vec{\nabla} f$ does not exist.

$\Rightarrow x = 0,\ y = 0$

$(0,0)$ is a critical point.

cannot use the Second derivative test.

$f(0,0) = 1$

$f(x,y) = 1 - \underbrace{\sqrt[3]{x^2+y^2}}_{\geq 0} \leq 1$ for all (x,y)

$\Rightarrow (0,0)$ is a local max.

Q23 $f(x,y) = y \sin x$

$$\begin{cases} f_x = y\cos x = 0 \\ f_y = \sin x = 0 \end{cases} \Rightarrow \begin{cases} x = n\pi, & n \in \mathbb{Z} \\ y = 0 \end{cases}$$

$f_{xx} = -y\sin x,\quad f_{yy} = 0,\quad f_{xy} = \cos x$

$f_{xx}(n\pi, 0) = 0,\quad f_{yy}(n\pi, 0) = 0$

$$f_{xy}(n\pi, 0) = \begin{cases} 1, & \text{if } n \text{ is even} \\ -1, & \text{if } n \text{ is odd} \end{cases}$$

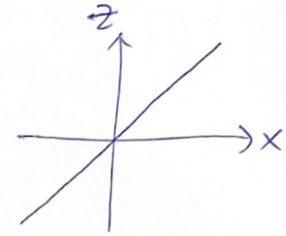
$f_{xx}f_{yy} - f_{xy}^2 = 0 - 1 = -1 < 0$

$\Rightarrow (n\pi, 0), n \in \mathbb{Z}$ are saddle points

Q44 (c) The discriminant $f_{xx}f_{yy} - f_{xy}^2$ is zero at the origin for the following function, so the Second Derivative Test fails there. Determine whether the function has a maximum, a minimum, or neither at the origin by imagining what the surface $z = f(x,y)$ looks like.

$f(x,y) = xy^2$

$f(0,0) = 0$

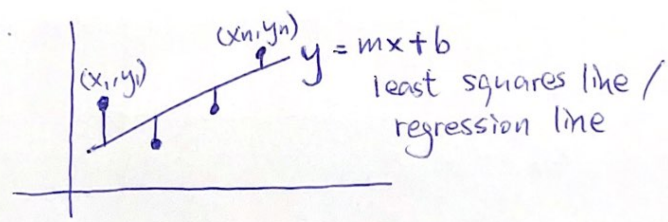


$f(x,y) < 0$ for $x < 0$

$f(x,y) > 0$ for $x > 0$

$\Rightarrow$ Neither

Q65 When we try to fit a line $y = mx+b$ to a set of numerical data points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$. We usually choose the line that minimizes the sum of the squares of the vertical distances from the points to the line.



i.e. finding m, b that minimize

$$w = (mx_1 + b - y_1)^2 + \cdots + (mx_n + b - y_n)^2.$$

Show that
$$m = \frac{\left(\sum_{k=1}^{n} x_k\right)\left(\sum_{k=1}^{n} y_k\right) - n\sum_{k=1}^{n} x_k y_k}{\left(\sum_{k=1}^{n} x_k\right)^2 - n\sum_{k=1}^{n} x_k^2}$$

$$b = \frac{1}{n}\left(\sum_{k=1}^{n} y_k - m\sum_{k=1}^{n} x_k\right)$$

$$w = (mx_1+b-y_1)^2 + \dots + (mx_n+b-y_n)^2$$

$$w_m = 2(mx_1+b-y_1)(x_1) + \dots + 2(mx_n+b-y_n)(x_n)$$

$$= 2\left((mx_1+b-y_1)(x_1) + \dots + (mx_n+b-y_n)(x_n)\right) = 0$$

$$\Rightarrow mx_1^2 + bx_1 - x_1y_1 + \dots + mx_n^2 + bx_n - x_ny_n = 0$$

$$\Rightarrow m(x_1^2+\dots+x_n^2) + b(x_1+\dots+x_n) - (x_1y_1+\dots+x_ny_n) = 0$$

$$\Rightarrow m\sum_{k=1}^{n} x_k^2 + b\sum_{k=1}^{n} x_k - \sum_{k=1}^{n} x_ky_k = 0 \qquad (*)$$

$$w_b = 2(mx_1+b-y_1)(1) + \dots + 2(mx_n+b-y_n)(1)$$

$$= 2\left((mx_1+b-y_1) + \dots + (mx_n+b-y_n)\right) = 0$$

$$\Rightarrow (mx_1+b-y_1) + \dots + (mx_n+b-y_n) = 0$$

$$\Rightarrow m\sum_{k=1}^{n} x_k + bn - \sum_{k=1}^{n} y_k = 0$$

$$\Rightarrow b = \frac{1}{n}\left(\sum_{k=1}^{n} y_k - m\sum_{k=1}^{n} x_k\right)$$

Sub b in $(*)$:

$$m\sum_{k=1}^{n} x_k^2 + \left(\frac{1}{n}\left(\sum_{k=1}^{n} y_k - m\sum_{k=1}^{n} x_k\right)\right)\sum_{k=1}^{n} x_k - \sum_{k=1}^{n} x_ky_k = 0$$

$$mn\sum_{k=1}^{n} x_k^2 + \left(\sum_{k=1}^{n} x_k\right)\left(\sum_{k=1}^{n} y_k\right) - m\left(\sum_{k=1}^{n} x_k\right)^2 - n\sum_{k=1}^{n} x_ky_k = 0.$$

$$\left(\sum_{k=1}^{n} x_k\right)\left(\sum_{k=1}^{n} y_k\right) - n\sum_{k=1}^{n} x_ky_k = m\left(\left(\sum_{k=1}^{n} x_k\right)^2 - n\sum_{k=1}^{n} x_k^2\right)$$

$$\Rightarrow m = \frac{\left(\sum_{k=1}^{n} x_k\right)\left(\sum_{k=1}^{n} y_k\right) - n\sum_{k=1}^{n} x_ky_k}{\left(\sum_{k=1}^{n} x_k\right)^2 - n\sum_{k=1}^{n} x_k^2}$$

Show this <u>minimize</u> w :

use second derivative test

$$w_{mm} = 2x_1^2 + \dots + 2x_n^2 = 2\sum_{k=1}^{n} x_k^2$$

$$w_{bb} = 2 + \dots + 2 = 2n$$

$$w_{mb} = 2x_1 + \dots + 2x_n = 2\sum_{k=1}^{n} x_k$$

- Discriminant $w_{mm} w_{bb} - w_{mb}^2 = \left(2\sum_{k=1}^{n} x_k^2\right)(2n) - \left(2\sum_{k=1}^{n} x_k\right)^2$

$$= 4\left(n\sum_{k=1}^{n} x_k^2 - \left(\sum_{k=1}^{n} x_k\right)^2\right)$$

$$n\sum_{k=1}^{n} x_k^2 - \left(\sum_{k=1}^{n} x_k\right)^2$$

$$= n(x_1^2 + x_2^2 + \dots + x_n^2) - (x_1 + x_2 + \dots + x_n)(x_1 + x_2 + \dots + x_n)$$

$$= nx_1^2 + \dots + nx_n^2 - x_1^2 - x_1x_2 - \dots - x_1x_n - x_2x_1 - x_2^2 - \dots - x_n^2$$

$$= (n-1)x_1^2 + \dots + (n-1)x_n^2 - 2x_1x_2 - 2x_1x_3 - \dots - 2x_{n-1}x_n$$

$$= (x_1^2 - 2x_1x_2 + x_2^2) + (x_1^2 - 2x_1x_3 + x_3^2) + \dots + (x_{n-1}^2 - 2x_{n-1}x_n + x_n^2)$$

$$= (x_1 - x_2)^2 + (x_1 - x_3)^2 + \dots + (x_{n-1} - x_n)^2$$

$$\geq 0$$

$\Rightarrow$ Discriminant ≥ 0

$x_1 = x_2 = \dots = x_n \iff$ Discriminant $= 0$

- $w_{mm} = 2\sum_{k=1}^{n} x_k^2 \geq 0$

$x_1 = x_2 = \dots = x_n = 0 \iff w_{mm} = 0$

$\therefore$ Provided that at least one $x_i \neq 0$ and $x_i \neq x_j$, $j \neq i$,

then Discriminant > 0 and $w_{mm} > 0$.

$\Rightarrow$ m, b <u>minimize</u> w.

Q 66 Find the least squares line for the set of data points.
Use the linear equation to predict the value of y that would correspond to x = 4.

(−2, 0), (0, 2), (2, 3)

	x_k	y_k	x_k^2	$x_k y_k$
	−2	0	4	0
	0	2	0	0
	2	3	4	6
Σ	0	5	8	6

$$m = \frac{0 \cdot 5 - 3(6)}{0^2 - 3(8)} = \frac{3}{4}$$

$$b = \frac{1}{3}\left(5 - \frac{3}{4}(0)\right) = \frac{5}{3}$$

$$\Rightarrow y = \frac{3}{4}x + \frac{5}{3}$$

When $x = 4$, $y = \frac{3}{4}(4) + \frac{5}{3} = \frac{14}{3}$