MATH2010 2023-24

Tutorial 7 Term 1

Exercise 14.6

Q2 Find equations for the

(a) tangent plane and

(b) normal line at the point P_0 on the given surface

$$\underbrace{x^2+y^2-z^2}_{f(x,y,z)} = 18 , \quad P_0(3,5,-4)$$

surface $f(x,y,z) = c$

Curve $\tilde{r}(t) = x(t)\,\tilde{i} + y(t)\,\tilde{j} + z(t)\,\tilde{k}$

$$\underset{\substack{\uparrow \\ \because f \text{ is constant}}}{0} = \frac{d}{dt} f(\tilde{r}(t)) = \nabla f(\tilde{r}(t)) \cdot \underset{\text{tangent vector}}{\tilde{r}'(t)} \qquad (\text{chain rule})$$

$$\boxed{\begin{array}{l} \nabla f \perp \text{ tangent plane} \\ \nabla f \text{ parallel to normal line.} \end{array}}$$

$f_x(P_0)(x-x_0) + f_y(P_0)(y-y_0) + f_z(P_0)(z-z_0) = 0$

$x = x_0 + f_x(P_0)t$,
$y = y_0 + f_y(P_0)t$,
$z = z_0 + f_z(P_0)t$

(a) $\nabla f = (2x, 2y, -2z)$

$\nabla f(3,5,-4) = (6, 10, 8)$

$\Rightarrow$ Tangent plane :

$$6(x-3) + 10(y-5) + 8(z+4) = 0$$

$$3x + 5y + 4z = 18$$

(b) Normal line :

$$x = 3 + 6t , \quad y = 5 + 10t , \quad z = -4 + 8t$$

Q9 Find an equation for the plane that is tangent to the given surface at the given point.

$$\underbrace{z = \ln(x^2+y^2)}_{f(x,y)} , \quad (1,0,0)$$

$$\boxed{f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) - (z-z_0) = 0}$$

$$f_x(x,y) = \frac{1}{x^2+y^2} \cdot 2x = \frac{2x}{x^2+y^2}$$

$$f_y(x,y) = \frac{1}{x^2+y^2} \cdot 2y = \frac{2y}{x^2+y^2}$$

$$f_x(1,0) = 2$$

$$f_y(1,0) = 0$$

Tangent plane :

$$2(x-1) + 0\cdot(y-0) - (z-0) = 0$$

$$2(x-1) - z = 0$$

$$2x - z = 2$$

$$\Big(\underbrace{\ln(x^2+y^2) - z}_{f(x,y,z)} = 0$$

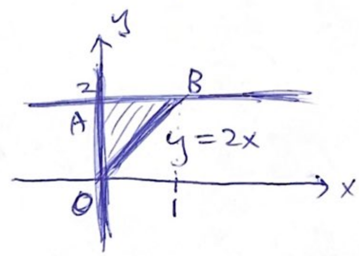
$$f_z = -1 \Big)$$

Exercise 14.7

Q31 Find the absolute maxima and minima of the functions on the given domains.

$f(x,y) = 2x^2 - 4x + y^2 - 4y + 1$

on the closed triangular plate bounded by the lines $x=0$, $y=2$, $y=2x$ in the first quadrant.



- Interior points :

$$\boxed{\begin{array}{l}\text{local maximum / minimum } (a,b): \\ f_x(a,b) = 0, \quad f_y(a,b) = 0\end{array}}$$

$f_x(x,y) = 4x - 4 \Rightarrow x = 1$

$f_y(x,y) = 2y - 4 \Rightarrow y = 2$

$(1,2)$: not an interior point

- Boundary points :

On OA : $x = 0$

$f(x,y) = y^2 - 4y + 1$, $0 \le y \le 2$

$f'(0,y) = 2y - 4 = 0 \Rightarrow y = 2$ endpoint $\underline{f(0,2) = -3}$

other endpoints : $\underline{f(0,0) = 1}$

On AB : $y = 2$ $0 \le x \le 1$

$f(x,y) = f(x,2) = 2x^2 - 4x + 2^2 - 4(2) + 1 = 2x^2 - 4x - 3$

$f'(x,2) = 4x - 4 = 0 \Rightarrow x = 1$ endpoint $\underline{f(1,2) = -5}$

Another endpoint value $f(0,2)$ already found above

On OB : $y = 2x$

$$f(x,y) = f(x,2x) = 2x^2 - 4x + (2x)^2 - 4(2x) + 1$$

$$= 6x^2 - 12x + 1 \quad , \quad 0 \le x \le 1$$

$$f'(x,2x) = 12x - 12 = 0 \Rightarrow x = 1 \qquad f(1,2) = -5$$

Absolute maximum $= 1$ at $(0,0)$

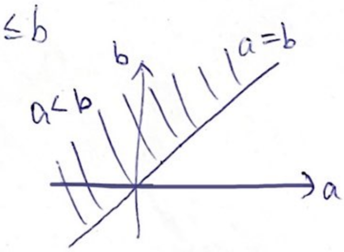
Absolute minimum $= -5$ at $(1,2)$.

Q40 Find two numbers a and b with $a \le b$ such that

$$\int_a^b (24 - 2x - x^2)^{\frac{1}{3}}\, dx$$

has its largest value.

Let $F(a,b) = \int_a^b (24 - 2x - x^2)^{\frac{1}{3}}\, dx \quad , \quad a \le b$



- Boundary : $a = b$

$$F(a,b) = 0$$

- Interior : $F_a = -(24 - 2a - a^2)^{\frac{1}{3}}$

$$F_b = (24 - 2b - b^2)^{\frac{1}{3}}$$

$\int f(x)\, dx = F(x) + C$

$\int_a^b f(x)\, dx = F(b) - F(a)$

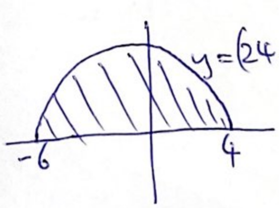
$F_a = 0 \Rightarrow a = -6, 4$

$F_b = 0 \Rightarrow b = -6, 4$

$a < b \Rightarrow (a,b) = (-6, 4)$

only one critical point

$F(-6,4) = \int_{-6}^{4} (24 - 2x - x^2)^{\frac{1}{3}}\, dx$



Area > 0

$\Rightarrow a = -6$

$b = 4$

Q61 Find the absolute maximum and minimum values of the following functions on the given curves :

a. $f(x,y) = x+y$ b. $g(x,y) = xy$ c. $h(x,y) = 2x^2+y^2$

Curves : i) The semicircle $x^2+y^2=4$, $y \geq 0$

ii) The quarter circle $x^2+y^2=4$, $x \geq 0, y \geq 0$

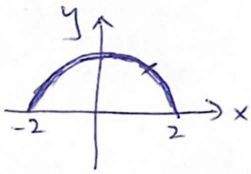
Use the parametric equations $x = 2\cos t$, $y = 2\sin t$.

(a) $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}$ (Chain rule)

$= -2\sin t + 2\cos t = 0 \Rightarrow \cos t = \sin t$

$\Rightarrow x = y$

i) interior :



$t = \frac{\pi}{4}$

$x = y = \sqrt{2}$

$f(\sqrt{2},\sqrt{2}) = 2\sqrt{2}$

endpoints :

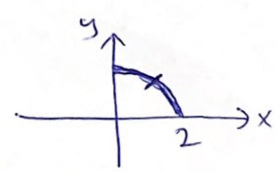
$f(-2,0) = -2$, $f(2,0) = 2$

Absolute maximum $= f(\sqrt{2},\sqrt{2}) = 2\sqrt{2}$ when $t = \frac{\pi}{4}$

Absolute minimum $= f(-2,0) = -2$ when $t = \pi$

ii) interior: $f(\sqrt{2},\sqrt{2}) = 2\sqrt{2}$, $t = \frac{\pi}{4}$

endpoints : $f(0,2) = 2$, $f(2,0) = 2$



Absolute maximum $= f(\sqrt{2},\sqrt{2}) = 2\sqrt{2}$ when $t = \frac{\pi}{4}$

Absolute minimum : $f(0,2) = 2$ when $t = \frac{\pi}{2}$

and $f(2,0) = 2$ when $t = 0$

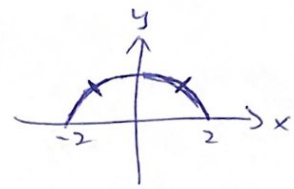
(b) $\frac{dg}{dt} = \frac{\partial g}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial g}{\partial y} \cdot \frac{dy}{dt}$

$= y \cdot (-2 \sin t) + x \cdot (2 \cos t)$

$= -4 \sin^2 t + 4 \cos^2 t = 0 \Rightarrow \cos t = \pm \sin t$

$\Rightarrow x = \pm y$

i) interior :



$x = y = \sqrt{2}$ at $t = \frac{\pi}{4}$

$x = -\sqrt{2}$, $y = \sqrt{2}$ at $t = \frac{3\pi}{4}$

$g(\sqrt{2}, \sqrt{2}) = 2$, $g(-\sqrt{2}, \sqrt{2}) = -2$

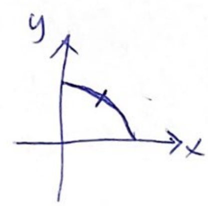
endpoints :

$g(-2, 0) = 0$, $g(2, 0) = 0$

Absolute maximum $= g(\sqrt{2}, \sqrt{2}) = 2$ when $t = \frac{\pi}{4}$

Absolute minimum $= g(-\sqrt{2}, \sqrt{2}) = -2$ when $t = \frac{3\pi}{4}$.

ii) interior :



$g(\sqrt{2}, \sqrt{2}) = 2$

endpoints :

$g(0, 2) = 0$, $g(2, 0) = 0$

Absolute maximum $= g(\sqrt{2}, \sqrt{2}) = 2$ when $t = \frac{\pi}{4}$.

Absolute minimum $= g(0, 2) = 0$ when $t = \frac{\pi}{2}$

and $g(2, 0) = 0$ when $t = 0$

(c) $\dfrac{dh}{dt} = \dfrac{\partial h}{\partial x} \cdot \dfrac{dx}{dt} + \dfrac{\partial h}{\partial y} \cdot \dfrac{dy}{dt}$

$= 4x \cdot (-2\sin t) + 2y\,(2\cos t)$

$= (8\cos t)(-2\sin t) + (4\sin t)(2\cos t)$

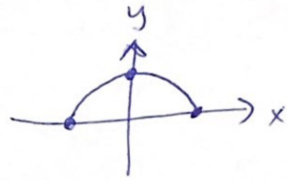
$= -8\cos t \sin t = 0 \Rightarrow -4\sin 2t = 0$

$\Rightarrow t = 0, \frac{\pi}{2}, \pi$

i) when $t = 0$, $h(2,0) = 8$

when $t = \frac{\pi}{2}$, $h(0,2) = 4$

when $t = \pi$, $h(-2,0) = 8$



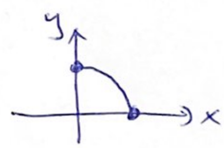
Absolute maximum : $h(2,0) = 8$ when $t = 0$

and $h(-2,0) = 8$ when $t = \pi$

Absolute minimum = $h(0,2) = 4$ when $t = \frac{\pi}{2}$

ii) Absolute maximum

$= h(2,0) = 8$ when $t = 0$



Absolute minimum

$= h(0,2) = 4$ when $t = \frac{\pi}{2}$