MATH2010 2023-24

Tutorial 6 Term 1

Exercise 14.5

Q 23 Find the directions in which the functions increase and decrease most rapidly at P_0. Then find the derivatives of the functions in these directions.

$$f(x,y,z) = \ln xy + \ln yz + \ln xz, \quad P_0(1,1,1)$$

$$\boxed{\text{At a point } \vec{a}, \ f \text{ increases (decreases) most rapidly in the direction of } \vec{\nabla} f(\vec{a}) \ (-\vec{\nabla} f(\vec{a})) \text{ at a rate of } \|\vec{\nabla} f(\vec{a})\|.}$$

$$\vec{\nabla} f(\vec{a}) = \left(\frac{\partial f}{\partial x_1}(\vec{a}), \ldots, \frac{\partial f}{\partial x_n}(\vec{a})\right)$$

$$\vec{\nabla} f(x,y,z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

$$= \left(\frac{1}{xy}\cdot y + 0 + \frac{1}{xz}\cdot z, \ \frac{1}{xy}\cdot x + \frac{1}{yz}\cdot z, \ \frac{1}{yz}\cdot y + \frac{1}{xz}\cdot x\right)$$

$$= \left(\frac{2}{x}, \frac{2}{y}, \frac{2}{z}\right)$$

$$\vec{\nabla} f(1,1,1) = (2,2,2), \quad \|\vec{\nabla} f(1,1,1)\| = 2\sqrt{3}$$

$$\frac{\vec{\nabla} f(1,1,1)}{\|\vec{\nabla} f(1,1,1)\|} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

Hence, f increases most rapidly in the direction $\vec{u} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ and decreases most rapidly in the direction $-\vec{u} = \left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$

$$D_{\vec{u}} f(1,1,1) = \vec{\nabla} f(1,1,1)\cdot \vec{u} = \|\vec{\nabla} f(1,1,1)\| = 2\sqrt{3}, \quad D_{-\vec{u}} f(1,1,1) = -2\sqrt{3}$$

Q29 Let $f(x,y) = x^2 - xy + y^2 - y$. Find the directions $\vec{u}$ and the values of $D_{\vec{u}} f(1,-1)$ for which

a. $D_{\vec{u}} f(1,-1)$ is largest
b. $D_{\vec{u}} f(1,-1)$ is smallest
c. $D_{\vec{u}} f(1,-1) = 0$
d. $D_{\vec{u}} f(1,-1) = 4$
e. $D_{\vec{u}} f(1,-1) = -3$

$$\boxed{D_{\vec{u}} f(\vec{a}) = \vec{\nabla} f(\vec{a}) \cdot \vec{u}}$$

↑ unit vector in $\mathbb{R}^n$

$$\vec{\nabla} f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)$$
$$= (2x-y, -x+2y-1)$$

$\vec{\nabla} f(1,-1) = (3,-4)$, $\|\vec{\nabla} f(1,-1)\| = 5$

a. increases most rapidly

direction $\vec{u} = \left(\frac{3}{5}, -\frac{4}{5}\right)$

$D_{\vec{u}} f(1,-1) = 5$

b. decreases most rapidly

direction $\vec{u} = \left(-\frac{3}{5}, \frac{4}{5}\right)$

$D_{\vec{u}} f(1,-1) = -5$

c. direction $\vec{u} = \left(\frac{4}{5}, \frac{3}{5}\right)$ or $\vec{u} = \left(-\frac{4}{5}, -\frac{3}{5}\right)$

d. Let $\vec{u} = (u_1, u_2)$, $\sqrt{u_1^2 + u_2^2} = 1 \Rightarrow u_1^2 + u_2^2 = 1$

$$D_{\vec{u}} f(1,-1) = 4$$

$$\vec{\nabla} f(1,-1) \cdot \vec{u} = 4$$

$$(3,-4) \cdot (u_1, u_2) = 4$$

$$3u_1 - 4u_2 = 4$$

$$\Rightarrow u_2 = \frac{3}{4} u_1 - 1$$

$$\Rightarrow u_1^2 + \left(\frac{3}{4}u_1 - 1\right)^2 = 1$$

$$\frac{25}{16} u_1^2 - \frac{3}{2} u_1 + 1 = 1$$

$$u_1 \left(\frac{25}{16} u_1 - \frac{3}{2}\right) = 0$$

$$u_1 = 0 \quad \text{or} \quad u_1 = \frac{24}{25}$$

For $u_1 = 0$, $u_2 = -1$, direction $\vec{u} = (0, -1)$

For $u_1 = \frac{24}{25}$, $u_2 = -\frac{7}{25}$, direction $\vec{u} = \left(\frac{24}{25}, -\frac{7}{25}\right)$

e. $3u_1 - 4u_2 = -3$

$$\Rightarrow u_1 = \frac{4}{3} u_2 - 1$$

$$\Rightarrow \left(\frac{4}{3}u_2 - 1\right)^2 + u_2^2 = 1$$

$$\frac{25}{9} u_2^2 - \frac{8}{3} u_2 + 1 = 1$$

$$u_2 \left(\frac{25}{9} u_2 - \frac{8}{3}\right) = 0$$

$$u_2 = 0 \quad \text{or} \quad u_2 = \frac{24}{25}$$

For $u_2 = 0$, $u_1 = -1$, direction $\vec{u} = (-1, 0)$

For $u_2 = \frac{24}{25}$, $u_1 = \frac{7}{25}$, direction $\vec{u} = \left(\frac{7}{25}, \frac{24}{25}\right)$

Exercise 14.6

Q 19 By about how much will

$$f(x,y,z) = \ln\sqrt{x^2+y^2+z^2}$$

change if the point $P(x,y,z)$ moves from $P_0(3,4,12)$ a distance of $ds = 0.1$ unit in the direction of $3\vec{i} + 6\vec{j} - 2\vec{k}$?

$$\boxed{df = (\vec{\nabla} f|_{P_0} \cdot \vec{u})\, ds}$$

$$f(x,y,z) = \frac{1}{2}\ln(x^2+y^2+z^2)$$

$$\vec{\nabla} f(x,y,z) = \left(\frac{1}{2}\cdot\frac{1}{x^2+y^2+z^2}\cdot 2x,\ \frac{1}{2}\cdot\frac{1}{x^2+y^2+z^2}\cdot 2y,\ \frac{1}{2}\cdot\frac{1}{x^2+y^2+z^2}\cdot 2z\right)$$

$$= \left(\frac{x}{x^2+y^2+z^2},\ \frac{y}{x^2+y^2+z^2},\ \frac{z}{x^2+y^2+z^2}\right)$$

$$\vec{\nabla} f(3,4,12) = \left(\frac{3}{169},\ \frac{4}{169},\ \frac{12}{169}\right)$$

$$\|3,6,-2\| = 7$$

$$\vec{u} = \left(\frac{3}{7},\ \frac{6}{7},\ -\frac{2}{7}\right)$$

$$df = \left(\left(\frac{3}{169},\ \frac{4}{169},\ \frac{12}{169}\right)\cdot\left(\frac{3}{7},\ \frac{6}{7},\ -\frac{2}{7}\right)\right)(0.1)$$

$$= \left(\frac{9}{1183}\right)(0.1) = 0.00076$$

Q33 Find the linearization $L(x,y)$ of the function $f(x,y)$ at P_0. Then find an upper bound for the magnitude $|E|$ of the error in the approximation $f(x,y) \approx L(x,y)$ over the rectangle R.

$f(x,y) = x^2 - 3xy + 5$ at $P_0(2,1)$,

$R: |x-2| \leq 0.1, \quad |y-1| \leq 0.1$

Linearization of $\vec{f}$ at $\vec{a}$

$$L(\vec{x}) = \vec{f}(\vec{a}) + \underbrace{D\vec{f}(\vec{a})}_{\begin{pmatrix} \vec{\nabla} f_1(\vec{a}) \\ \vdots \\ \vec{\nabla} f_m(\vec{a}) \end{pmatrix}} (\vec{x} - \vec{a})$$

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

$f(2,1) = 3$

$f_x(x,y) = 2x - 3y, \quad f_y(x,y) = -3x$

$f_x(2,1) = 1, \quad f_y(2,1) = -6$

$L(x,y) = 3 + 1(x-2) - 6(y-1) = 7 + x - 6y$

$|E(x,y)| \leq \frac{1}{2} M(|x-x_0| + |y-y_0|)^2$
where M is an upper bound for the values of $|f_{xx}|$, $|f_{yy}|$ and $|f_{xy}|$ on R.

$f_{xx} = 2, \quad f_{yy} = 0, \quad f_{xy} = -3 \quad \Rightarrow M = 3$

$|E| \leq \frac{1}{2}(3)(0.1 + 0.1)^2 = 0.06$

Q39 Find the linearizations $L(x,y,z)$ of the functions at the given points.

$f(x,y,z) = xy + yz + xz$ at

a. $(1,1,1)$ b. $(1,0,0)$ c. $(0,0,0)$

$f_x = y + z$
$f_y = x + z$
$f_z = y + x$

(a) $f(1,1,1) = 3$

$f_x(1,1,1) = 2, \quad f_y(1,1,1) = 2, \quad f_z(1,1,1) = 2$

$L(x,y,z) = 3 + 2(x-1) + 2(y-1) + 2(z-1)$

(b) $f(1,0,0) = 0$

$f_x(1,0,0) = 0, \quad f_y(1,0,0) = 1, \quad f_z(1,0,0) = 1$

$L(x,y,z) = 0 + 0(x-1) + 1(y-0) + 1(z-0)$

$= y + z$

(c) $f(0,0,0) = 0$

$f_x(0,0,0) = 0, \quad f_y(0,0,0) = 0, \quad f_z(0,0,0) = 0$

$L(x,y,z) = 0$