

## Chapter 14 Additional and Advanced Exercises (P. 879)

Q3 Leibniz's Rule:

If  $f$  is continuous on  $[a, b]$  and if  $u(x)$  and  $v(x)$  are differentiable functions of  $x$  whose values lie in  $[a, b]$ , then

$$\frac{d}{dx} \int_{u(x)}^{v(x)} f(t) dt = f(v(x)) \frac{dv}{dx} - f(u(x)) \frac{du}{dx}.$$

Prove the rule by setting

$$g(u, v) = \int_u^v f(t) dt, \quad u = u(x), \quad v = v(x)$$

and calculating  $\frac{dg}{dx}$  with the Chain Rule.

$$\frac{dg}{dx} = \frac{\partial g}{\partial u} \frac{du}{dx} + \frac{\partial g}{\partial v} \frac{dv}{dx} \quad (\text{Chain Rule})$$

$$= \left( \frac{\partial}{\partial u} \int_u^v f(t) dt \right) \frac{du}{dx} + \left( \frac{\partial}{\partial v} \int_u^v f(t) dt \right) \frac{dv}{dx}$$

$$= \left( - \frac{\partial}{\partial u} \int_u^v f(t) dt \right) \frac{du}{dx} + \left( \frac{\partial}{\partial v} \int_u^v f(t) dt \right) \frac{dv}{dx}$$

$$= -f(u) \frac{du}{dx} + f(v) \frac{dv}{dx}$$

$$= f(v(x)) \frac{dv}{dx} - f(u(x)) \frac{du}{dx}$$

Q4 Suppose that  $f$  is a twice-differentiable function of  $r$ , that  $r = \sqrt{x^2 + y^2 + z^2}$ , and that

$$f_{xx} + f_{yy} + f_{zz} = 0.$$

Show that for some constants  $a$  and  $b$ ,

$$f(r) = \frac{a}{r} + b.$$

By chain rule,

$$f_x = \frac{df}{dr} \frac{\partial r}{\partial x}$$

$$\begin{aligned} f_{xx} &= \left(\frac{d^2f}{dr^2}\right) \frac{\partial r}{\partial x} \cdot \frac{\partial r}{\partial x} + \frac{df}{dr} \frac{\partial^2 r}{\partial x^2} \\ &= \left(\frac{d^2f}{dr^2}\right) \left(\frac{\partial r}{\partial x}\right)^2 + \frac{df}{dr} \frac{\partial^2 r}{\partial x^2} \end{aligned}$$

$$\text{Similarly, } f_{yy} = \left(\frac{d^2f}{dr^2}\right) \left(\frac{\partial r}{\partial y}\right)^2 + \frac{df}{dr} \frac{\partial^2 r}{\partial y^2}$$

$$f_{zz} = \left(\frac{d^2f}{dr^2}\right) \left(\frac{\partial r}{\partial z}\right)^2 + \frac{df}{dr} \frac{\partial^2 r}{\partial z^2}$$

$$\frac{\partial r}{\partial x} = \frac{1}{2} \frac{1}{\sqrt{x^2 + y^2 + z^2}} (2x) = \frac{x}{\sqrt{x^2 + y^2 + z^2}}$$

$$\frac{\partial^2 r}{\partial x^2} = \frac{1 \cdot \sqrt{x^2 + y^2 + z^2} - x \cdot \frac{x}{\sqrt{x^2 + y^2 + z^2}}}{x^2 + y^2 + z^2} = \frac{y^2 + z^2}{(\sqrt{x^2 + y^2 + z^2})^3}$$

$$\text{Similarly, } \frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2 + y^2 + z^2}}, \quad \frac{\partial^2 r}{\partial y^2} = \frac{x^2 + z^2}{(\sqrt{x^2 + y^2 + z^2})^3}$$

$$\frac{\partial r}{\partial z} = \frac{z}{\sqrt{x^2 + y^2 + z^2}}, \quad \frac{\partial^2 r}{\partial z^2} = \frac{x^2 + y^2}{(\sqrt{x^2 + y^2 + z^2})^3}$$

$$f_{xx} + f_{yy} + f_{zz} = 0$$

$$\Rightarrow \left(\frac{d^2f}{dr^2}\right) \frac{x^2}{x^2 + y^2 + z^2} + \left(\frac{df}{dr}\right) \frac{y^2 + z^2}{(\sqrt{x^2 + y^2 + z^2})^3}$$

$$+ \left(\frac{d^2f}{dr^2}\right) \frac{y^2}{x^2 + y^2 + z^2} + \left(\frac{df}{dr}\right) \frac{x^2 + z^2}{(\sqrt{x^2 + y^2 + z^2})^3}$$

$$+ \left(\frac{d^2f}{dr^2}\right) \frac{z^2}{x^2 + y^2 + z^2} + \left(\frac{df}{dr}\right) \frac{x^2 + y^2}{(\sqrt{x^2 + y^2 + z^2})^3} = 0$$

$$\Rightarrow \left( \frac{d^2 f}{dr^2} \right) + \left( \frac{df}{dr} \right) \left( \frac{2}{\sqrt{x^2 + y^2 + z^2}} \right) = 0$$

$$\frac{d^2 f}{dr^2} + \frac{2}{r} \frac{df}{dr} = 0$$

$$\frac{df'}{dr} = -\frac{2}{r} f' \quad , \quad \text{where } f' = \frac{df}{dr}$$

$$\int \frac{1}{f'} df' = \int -\frac{2}{r} dr$$

$$\ln f' = -2 \ln r + C'$$

$$f' = e^{-2 \ln r + C'} = Cr^{-2} \quad \text{for some constant } C, \quad C = e^{C'}$$

$$\Rightarrow f = \int Cr^{-2} dr$$

$$= C \frac{r^{-1}}{-1} + b \quad \text{for some constant } b$$

$$= \frac{a}{r} + b \quad \text{for some constants } a, b$$

$$a = -C$$

Q5 A function  $f(x, y)$  is homogeneous of degree  $n$  ( $n$  a nonnegative integer) if  $f(tx, ty) = t^n f(x, y)$  for all  $t, x$ , and  $y$ . For such a function (sufficiently differentiable), prove that

$$a. \quad x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = n f(x, y)$$

$$b. \quad x^2 \left( \frac{\partial^2 f}{\partial x^2} \right) + 2xy \left( \frac{\partial^2 f}{\partial x \partial y} \right) + y^2 \left( \frac{\partial^2 f}{\partial y^2} \right) = n(n-1) f$$

Q. Let  $u = tx$ ,  $v = ty$ ,

$$w = f(u, v) = f(tx, ty) = t^n f(x, y).$$

$$\frac{\partial w}{\partial t} = n t^{n-1} f(x, y)$$

||

$$\frac{\partial w}{\partial u} \frac{\partial u}{\partial t} + \frac{\partial w}{\partial v} \frac{\partial v}{\partial t} = \frac{\partial w}{\partial u} \cdot x + \frac{\partial w}{\partial v} \cdot y$$

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial w}{\partial v} \frac{\partial v}{\partial x} = \frac{\partial w}{\partial u} \cdot t + \frac{\partial w}{\partial v} \cdot 0$$

$$\Rightarrow \frac{\partial w}{\partial u} = \frac{1}{t} \frac{\partial w}{\partial x}$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial w}{\partial v} \frac{\partial v}{\partial y} = \frac{\partial w}{\partial u} \cdot 0 + \frac{\partial w}{\partial v} \cdot t$$

$$\Rightarrow \frac{\partial w}{\partial v} = \frac{1}{t} \frac{\partial w}{\partial y}$$

$$\begin{aligned} \Rightarrow n t^{n-1} f(x, y) &= \frac{1}{t} \frac{\partial w}{\partial x} \cdot x + \frac{1}{t} \frac{\partial w}{\partial y} \cdot y \\ &= \frac{x}{t} \frac{\partial w}{\partial x} + \frac{y}{t} \frac{\partial w}{\partial y} \end{aligned}$$

When  $t=1$ ,  $u=x$ ,  $v=y$ ,  $w=f(x, y)$ ,

$$\Rightarrow n f(x, y) = x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y}$$



b. From (a),  $n t^{n-1} f(x, y) = x \frac{\partial w}{\partial u} + y \frac{\partial w}{\partial v}$

Differentiate with respect to  $t$  again,

$$\begin{aligned} n(n-1) t^{n-2} f(x, y) &= x \left( \frac{\partial^2 w}{\partial u^2} \frac{\partial u}{\partial t} + \frac{\partial^2 w}{\partial v \partial u} \frac{\partial v}{\partial t} \right) \\ &\quad + y \left( \frac{\partial^2 w}{\partial u \partial v} \frac{\partial u}{\partial t} + \frac{\partial^2 w}{\partial v^2} \frac{\partial v}{\partial t} \right) \\ &= x^2 \frac{\partial^2 w}{\partial u^2} + 2xy \frac{\partial^2 w}{\partial u \partial v} + y^2 \frac{\partial^2 w}{\partial v^2} \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 w}{\partial x^2} &= \frac{\partial}{\partial x} \left( \frac{\partial w}{\partial x} \right) \stackrel{(a)}{=} \frac{\partial}{\partial x} \left( t \frac{\partial w}{\partial u} \right) = t \left( \frac{\partial^2 w}{\partial u^2} \frac{\partial u}{\partial x} + \frac{\partial^2 w}{\partial v \partial u} \frac{\partial v}{\partial x} \right) \\ &= t^2 \frac{\partial^2 w}{\partial u^2} \end{aligned}$$

Similarly,  $\frac{\partial^2 w}{\partial y^2} = t^2 \frac{\partial^2 w}{\partial v^2}$

$$\frac{\partial^2 w}{\partial y \partial x} = t^2 \frac{\partial^2 w}{\partial v \partial u}$$

$$\Rightarrow n(n-1) t^{n-2} f(x, y) = \frac{x^2}{t^2} \frac{\partial^2 w}{\partial x^2} + \frac{2xy}{t^2} \frac{\partial^2 w}{\partial y \partial x} + \frac{y^2}{t^2} \frac{\partial^2 w}{\partial y^2}, \quad t \neq 0$$

When  $t=1$ ,  $w = f(x, y)$ ,

$$\Rightarrow n(n-1) f(x, y) = x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2}$$

Q7 Properties of position vectors

Let  $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$  and let  $r = |\vec{r}|$ .

a. Show that  $\nabla r = \frac{\vec{r}}{r}$ .

b. Show that  $\nabla(r^n) = nr^{n-2}\vec{r}$ .

c. Find a function whose gradient equals  $\vec{r}$ .

d. Show that  $\vec{r} \cdot d\vec{r} = r dr$

e. Show that  $\nabla(\vec{A} \cdot \vec{r}) = \vec{A}$  for any constant vector  $\vec{A}$ .

a.  $r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$   
 $\nabla r = \frac{x}{\sqrt{x^2 + y^2 + z^2}} \vec{i} + \frac{y}{\sqrt{x^2 + y^2 + z^2}} \vec{j} + \frac{z}{\sqrt{x^2 + y^2 + z^2}} \vec{k} = \frac{\vec{r}}{r}$

b.  $r^n = (x^2 + y^2 + z^2)^{\frac{n}{2}}$   
 $\nabla(r^n) = nx(x^2 + y^2 + z^2)^{\frac{n}{2}-1} \vec{i} + ny(x^2 + y^2 + z^2)^{\frac{n}{2}-1} \vec{j} + nz(x^2 + y^2 + z^2)^{\frac{n}{2}-1} \vec{k} = nr^{n-2} \vec{r}$

c. Let  $n=2$  in (b),

$$\nabla(r^2) = 2\vec{r}$$

$$\Rightarrow \nabla\left(\frac{1}{2}r^2\right) = \vec{r}$$

$\frac{1}{2}r^2 = \frac{1}{2}(x^2 + y^2 + z^2)$  is the function.

d.  $d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

$$\vec{r} \cdot d\vec{r} = x dx + y dy + z dz$$

$$dr = r_x dx + r_y dy + r_z dz = \frac{x}{r} dx + \frac{y}{r} dy + \frac{z}{r} dz$$

$$r dr = x dx + y dy + z dz$$

e. Let  $\vec{A} = a\vec{i} + b\vec{j} + c\vec{k}$ .  $\vec{A} \cdot \vec{r} = ax + by + cz$

$$\Rightarrow \nabla(\vec{A} \cdot \vec{r}) = a\vec{i} + b\vec{j} + c\vec{k} = \vec{A}$$