MATH 2010 2023-24

Tutorial 10 Term 1

Exercise 14.8 Lagrange Multipliers

Three independent variables with 1 constraint

Finding extrema of $f(x,y,z)$

under constraint $g(x,y,z) = c$

Consider $\boxed{F(x,y,z,\lambda) = f(x,y,z) - \lambda(g(x,y,z)-c)}$

$$\begin{cases} \dfrac{\partial F}{\partial x} = 0 \\ \dfrac{\partial F}{\partial y} = 0 \\ \dfrac{\partial F}{\partial z} = 0 \\ \dfrac{\partial F}{\partial \lambda} = 0 \end{cases}$$

Q25 Find three real numbers whose sum is 9 and the sum of whose squares is as small as possible.

minimize $f(x,y,z) = x^2 + y^2 + z^2$

under constraint $g(x,y,z) = x+y+z = 9$

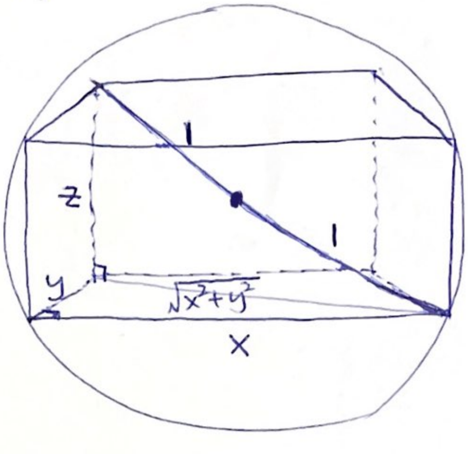
Consider $F(x,y,z,\lambda) = x^2+y^2+z^2 - \lambda(x+y+z-9)$

$$\begin{cases} \dfrac{\partial F}{\partial x} = 2x - \lambda = 0 \\ \dfrac{\partial F}{\partial y} = 2y - \lambda = 0 \\ \dfrac{\partial F}{\partial z} = 2z - \lambda = 0 \end{cases} \Rightarrow \begin{cases} 2x = \lambda \\ 2y = \lambda \\ 2z = \lambda \end{cases} \Rightarrow x = y = z$$

$$\dfrac{\partial F}{\partial \lambda} = -(x+y+z-9) = 0 \Rightarrow 3x - 9 = 0 \Rightarrow x = 3$$

$\Rightarrow x = 3,\ y = 3,\ z = 3.$

Q27 Find the dimensions of the closed rectangular box with maximum volume that can be inscribed in the unit sphere. (radius = 1)



maximize $V(x,y,z) = xyz$

under constraint $g(x,y,z)$

$= x^2+y^2+z^2 = 2^2 = 4$

Consider $F(x,y,z,\lambda)$

$= xyz - \lambda(x^2+y^2+z^2-4)$

$$\begin{cases} \frac{\partial F}{\partial x} = yz - 2\lambda x = 0 \\ \frac{\partial F}{\partial y} = xz - 2\lambda y = 0 \\ \frac{\partial F}{\partial z} = xy - 2\lambda z = 0 \\ \frac{\partial F}{\partial \lambda} = -(x^2+y^2+z^2-4) = 0 \end{cases} \Rightarrow \begin{cases} yz = 2\lambda x \\ xz = 2\lambda y \\ xy = 2\lambda z \end{cases}$$

$$\Rightarrow \begin{cases} xyz = 2\lambda x^2 \\ xyz = 2\lambda y^2 \\ xyz = 2\lambda z^2 \end{cases}$$

$\Rightarrow x^2 = y^2 = z^2$

$\Rightarrow 3x^2 = 4$

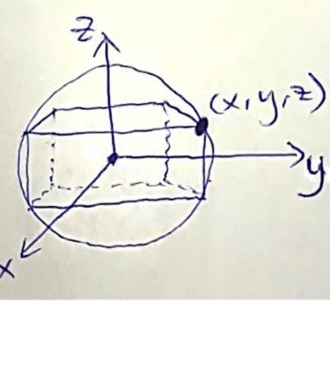
$x = \frac{2}{\sqrt{3}} \quad (x>0)$

$\Rightarrow x = \frac{2}{\sqrt{3}},\ y = \frac{2}{\sqrt{3}},\ z = \frac{2}{\sqrt{3}} \quad (x,y,z>0)$

$\therefore$ The dimensions of the rectangular box are $\frac{2}{\sqrt{3}}$ by $\frac{2}{\sqrt{3}}$ by $\frac{2}{\sqrt{3}}$.

Maximum volume $= \frac{8}{3\sqrt{3}}$.

(or



Maximize $V(x,y,z) = (2x)(2y)(2z)$

$= 8xyz$

Under Constraint $g(x,y,z)$

$= x^2+y^2+z^2 = 1$)

with <u>multiple constraints</u>

- Finding extrema of $f(\vec{x})$ (with $\vec{\nabla} g_i$ linearly independent) under constraints $g_i(\vec{x}) = c_i$ for $i = 1, \dots, k$.

Consider $\boxed{F(\vec{x}, \lambda_1, \dots, \lambda_k) = f(\vec{x}) - \sum_{i=1}^{k} \lambda_i (g_i(\vec{x}) - c_i)}$

$$\begin{cases} \dfrac{\partial F}{\partial x_j} = 0 & \forall j = 1, \dots, n \\ \dfrac{\partial F}{\partial \lambda_i} = 0 & \forall i = 1, \dots, k \end{cases}$$

Q37 Maximize the function $f(x,y,z) = x^2 + 2y - z^2$ subject to the constraints $\underbrace{2x - y}_{g_1(x,y,z)} = 0$ and $\underbrace{y + z}_{g_2(x,y,z)} = 0$

linearly independent $\rightarrow$ $\vec{\nabla} g_1 = (2, -1, 0)$, $\vec{\nabla} g_2 = (0, 1, 1)$

Consider $F(x, y, z, \lambda_1, \lambda_2) = x^2 + 2y - z^2 - \lambda_1 (2x - y) - \lambda_2 (y + z)$

$$\begin{cases} \dfrac{\partial F}{\partial x} = 2x - 2\lambda_1 = 0 \\ \dfrac{\partial F}{\partial y} = 2 + \lambda_1 - \lambda_2 = 0 \\ \dfrac{\partial F}{\partial z} = -2z - \lambda_2 = 0 \\ \dfrac{\partial F}{\partial \lambda_1} = -(2x - y) = 0 \\ \dfrac{\partial F}{\partial \lambda_2} = -(y + z) = 0 \end{cases} \Rightarrow \begin{cases} 2x = 2\lambda_1 \Rightarrow x = \lambda_1 \\ 2 = \lambda_2 - \lambda_1 \\ -2z = \lambda_2 \end{cases}$$

$\Rightarrow 2 = (-2z) - x$

$x = -2z - 2$

$\Rightarrow 2(-2z - 2) - y = 0$

$-y - 4z = 4$

$$\begin{cases} -y - 4z = 4 \\ y + z = 0 \end{cases}$$

$\Rightarrow z = -\frac{4}{3}$, $y = \frac{4}{3}$ $\Rightarrow x = \frac{2}{3}$

$\therefore (x, y, z) = (\frac{2}{3}, \frac{4}{3}, -\frac{4}{3})$

with maximum value $f(\frac{2}{3}, \frac{4}{3}, -\frac{4}{3}) = \frac{4}{3}$.

Q44 Find the point closest to the origin on the curve of intersection of the plane $2y+4z=5$ and the cone $z^2=4x^2+4y^2$

minimize $f(x,y,z)=x^2+y^2+z^2$ (square of distance from the origin)

under constraints $g_1(x,y,z) = 2y+4z=5$,

$g_2(x,y,z) = 4x^2+4y^2-z^2=0$.

$\vec{\nabla} g_1 = (0, 2, 4)$, $\vec{\nabla} g_2 = (8x, 8y, -2z)$

linearly independent $\Leftrightarrow$ $x \neq 0$ or $\frac{-2z}{8y} \neq \frac{4}{2}$

$\frac{z}{y} \neq -8$

Consider $F(x,y,z,\lambda_1,\lambda_2) = x^2+y^2+z^2 - \lambda_1(2y+4z-5) - \lambda_2(4x^2+4y^2-z^2)$

$$\begin{cases} \frac{\partial F}{\partial x} = 2x - 8\lambda_2 x = 0 \\ \frac{\partial F}{\partial y} = 2y - 2\lambda_1 - 8\lambda_2 y = 0 \Rightarrow \\ \frac{\partial F}{\partial z} = 2z - 4\lambda_1 + 2\lambda_2 z = 0 \\ \frac{\partial F}{\partial \lambda_1} = -(2y+4z-5) = 0 \\ \frac{\partial F}{\partial \lambda_2} = -(4x^2+4y^2-z^2) = 0 \end{cases} \qquad \begin{cases} 2x = 8\lambda_2 x \Rightarrow x=0 \text{ or } \lambda_2 = \frac{1}{4} \\ 2y = 2\lambda_1 + 8\lambda_2 y \\ 2z = 4\lambda_1 - 2\lambda_2 z \end{cases}$$

Case 1 : $x=0 \Rightarrow 4y^2 - z^2 = 0 \Rightarrow z = \pm 2y$

$\Rightarrow 2y + 4(2y) - 5 = 0$ or $2y + 4(-2y) - 5 = 0$

$y = \frac{1}{2}$ $\qquad y = -\frac{5}{6}$

$\Rightarrow (x,y,z) = (0, \frac{1}{2}, 1)$ or $(0, -\frac{5}{6}, \frac{5}{3})$

Case 2 : $\lambda_2 = \frac{1}{4} \Rightarrow y = \lambda_1 + y \Rightarrow \lambda_1 = 0$

$\Rightarrow 2z = 4(0) - 2(\frac{1}{4})z \Rightarrow z = 0 \Rightarrow y = \frac{5}{2}$

$\Rightarrow 4x^2 + 4(\frac{5}{2})^2 - 0^2 = 0 \Rightarrow$ no solutions

$f(0, \frac{1}{2}, 1) = \frac{5}{4}$

$f(0, -\frac{5}{6}, \frac{5}{3}) = \frac{125}{36}$

$\therefore$ $(0, \frac{1}{2}, 1)$ is closest to the origin.

Q45 The condition $\nabla f = \lambda \nabla g$ is not sufficient.

Although $\nabla f = \lambda \nabla g$ is a necessary condition for the occurrence of an extreme value of $f(x,y)$ subject to the conditions $g(x,y) = 0$ and $\nabla g \neq 0$, it does not in itself guarantee that one exists.

As a case in point, try using the method of Lagrange multipliers to find a maximum value of $f(x,y) = x+y$ subject to the constraint that $xy = 16$.

The method will identify the two points $(4,4)$ and $(-4,-4)$ as candidates for the location of extreme values.

Yet the sum $(x+y)$ has no maximum value on the hyperbola $xy = 16$.

The farther you go from the origin on this hyperbola in the first quadrant, the larger the sum $f(x,y) = x+y$ becomes. (When $x \to \infty, y \to 0, f(x,y) \to \infty$. When $x \to -\infty, y \to 0, f(x,y) \to -\infty$.)

No maximum / minimum

Consider $F(x, y, \lambda) = x + y - \lambda(xy - 16)$

$$\begin{cases} \frac{\partial F}{\partial x} = 1 - \lambda y = 0 \\ \frac{\partial F}{\partial y} = 1 - \lambda x = 0 \end{cases} \Rightarrow x = y$$

$\frac{\partial F}{\partial \lambda} = -(xy - 16) = 0 \Rightarrow y^2 = 16$

$\therefore (4,4), (-4,-4)$