MATH2010 Advanced Calculus I

Solution to Homework 1

§12.3 **Q**6

Solution.

a. $\mathbf{v}\cdot\mathbf{u} = (-1)(\sqrt{2}) + (1)(\sqrt{3}) + (0)(2) = \sqrt{3} - \sqrt{2}$
$|\mathbf{v}| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$
$|\mathbf{u}| = \sqrt{(\sqrt{2})^2 + (\sqrt{3})^2 + 2^2} = \sqrt{9} = 3$

b. $\cos\theta = \dfrac{\mathbf{v}\cdot\mathbf{u}}{|\mathbf{v}||\mathbf{u}|} = \dfrac{\sqrt{3}-\sqrt{2}}{(\sqrt{2})(3)} = \dfrac{\sqrt{6}-2}{6}$

c. $|\mathbf{u}|\cos\theta = 3\left(\dfrac{\sqrt{6}-2}{6}\right) = \dfrac{\sqrt{6}-2}{2}$

d. $\text{proj}_{\mathbf{v}}\mathbf{u} = (|\mathbf{u}|\cos\theta)\dfrac{\mathbf{v}}{|\mathbf{v}|} = \left(\dfrac{\sqrt{6}-2}{2}\right)\dfrac{-\mathbf{i}+\mathbf{j}}{\sqrt{2}} = \dfrac{\sqrt{2}-\sqrt{3}}{2}\mathbf{i} + \dfrac{\sqrt{3}-\sqrt{2}}{2}\mathbf{j}$

□

§12.3 **Q**13

Solution.

$\overrightarrow{AB} = \langle 2-(-1), 1-0\rangle = \langle 3, 1\rangle$
$\overrightarrow{AC} = \langle 1-(-1), -2-0\rangle = \langle 2, -2\rangle$
$\angle A = \cos^{-1}\dfrac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{|\overrightarrow{AB}||\overrightarrow{AC}|} = \cos^{-1}\dfrac{(3)(2)+(1)(-2)}{(\sqrt{3^2+1^2})(\sqrt{2^2+(-2)^2})} = \cos^{-1}\dfrac{4}{\sqrt{10}\sqrt{8}} = \cos^{-1}\dfrac{1}{\sqrt{5}} = 1.11 \text{ rad}$

$\overrightarrow{BC} = \langle 1-2, -2-1\rangle = \langle -1, -3\rangle$
$\overrightarrow{BA} = -\overrightarrow{AB} = \langle -3, -1\rangle$
$\angle B = \cos^{-1}\dfrac{\overrightarrow{BC}\cdot\overrightarrow{BA}}{|\overrightarrow{BC}||\overrightarrow{BA}|} = \cos^{-1}\dfrac{(-1)(-3)+(-3)(-1)}{(\sqrt{10})(\sqrt{10})} = \cos^{-1}\dfrac{6}{10} = \cos^{-1}\dfrac{3}{5} = 0.93 \text{ rad}$

$\angle C = \pi - (\angle A + \angle B) = \pi - \cos^{-1}\dfrac{1}{\sqrt{5}} - \cos^{-1}\dfrac{3}{5} = 1.11 \text{ rad}$

□

§12.3 **Q**28

Solution.

No, the same rule does not hold for dot product.
Let $\mathbf{u} = \langle 1, 1\rangle \neq \mathbf{0}, \mathbf{v_1} = \langle 1, 0\rangle, \mathbf{v_2} = \langle 0, 1\rangle$.
Then, $\mathbf{u}\cdot\mathbf{v_1} = 1, \mathbf{u}\cdot\mathbf{v_2} = 1$.
Hence, $\mathbf{u}\cdot\mathbf{v_1} = \mathbf{u}\cdot\mathbf{v_2}$ but $\mathbf{v_1} \neq \mathbf{v_2}$. □

§12.4 **Q2**

Solution.

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 0 \\ -1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 3 & 0 \\ 1 & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 0 \\ -1 & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix}\mathbf{k} = 0\mathbf{i} - 0\mathbf{j} + 5\mathbf{k} = 5\mathbf{k}$$

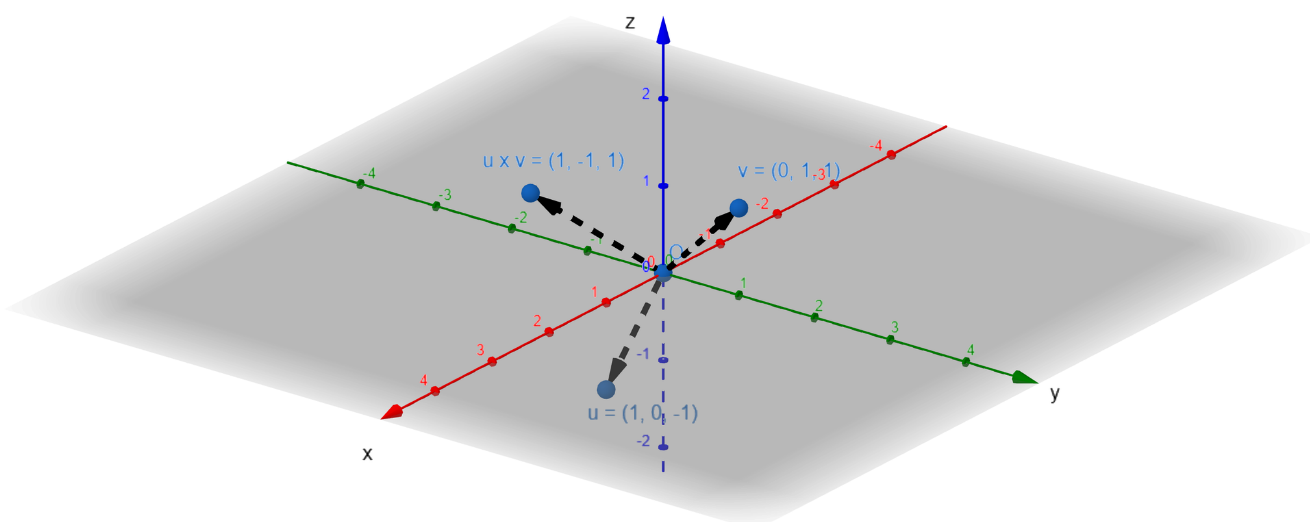
For $\mathbf{u}\times\mathbf{v}$, the length is 5 and direction is $\mathbf{k}$.
$\mathbf{v}\times\mathbf{u} = -(\mathbf{u}\times\mathbf{v}) = -5\mathbf{k}$
For $\mathbf{v}\times\mathbf{u}$, the length is 5 and direction is $-\mathbf{k}$.

□

§12.4 **Q11**

Solution.

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & -1 \\ 1 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}\mathbf{k} = \mathbf{i} - \mathbf{j} + \mathbf{k}$$



□

§12.4 **Q21**

Solution.

$$(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = \begin{vmatrix} 2 & 1 & 0 \\ 2 & -1 & 1 \\ 1 & 0 & 2 \end{vmatrix} = (2)(-2) - (1)(4-1) = -7$$

$$(\mathbf{v}\times\mathbf{w})\cdot\mathbf{u} = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 0 & 2 \\ 2 & 1 & 0 \end{vmatrix} = (2)(-2) - (1)(4-1) = -7$$

$$(\mathbf{w}\times\mathbf{u})\cdot\mathbf{v} = \begin{vmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 2 & -1 & 1 \end{vmatrix} = (1)(1) + (2)(-2-2) = -7$$

Hence, $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = (\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u} = (\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v}$.
Volume $= |(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}| = 7$ □

§12.5 **Q3**

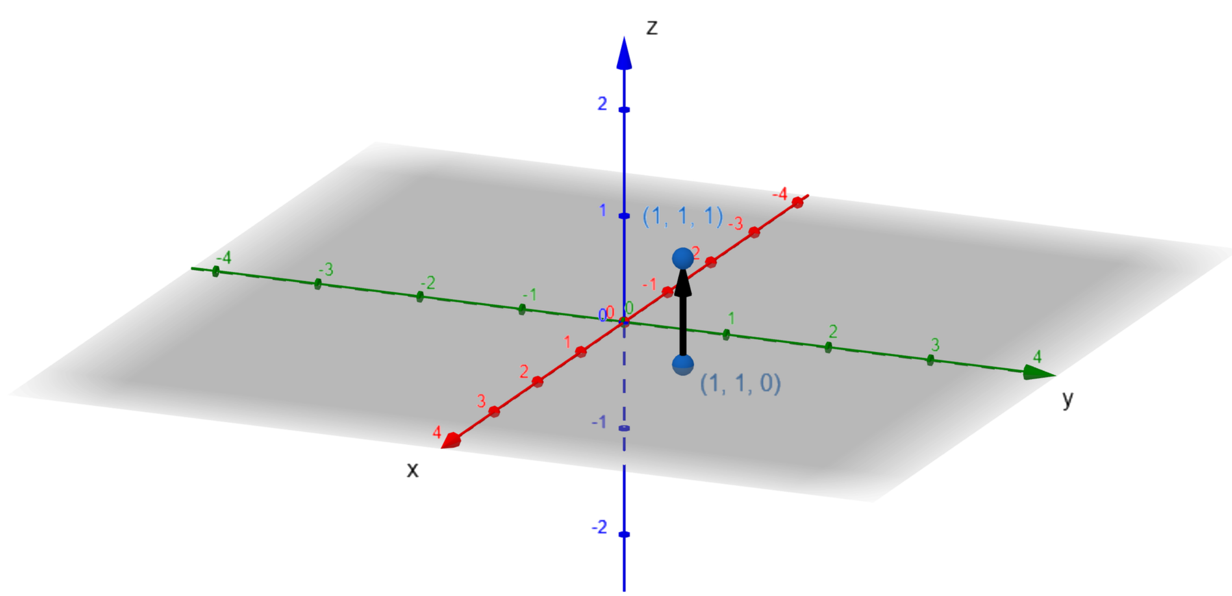
Solution.

$\overrightarrow{PQ} = (3 - (-2))\mathbf{i} + (5 - 0)\mathbf{j} + (-2 - 3)\mathbf{k} = 5\mathbf{i} + 5\mathbf{j} - 5\mathbf{k} = 5(\mathbf{i} + \mathbf{j} - \mathbf{k})$
Hence, the line is parallel to $\mathbf{i} + \mathbf{j} - \mathbf{k}$.
Since the line is passing through $P(-2, 0, 3)$, a parametrization of the line is
$x = -2 + t, y = t, z = 3 - t, -\infty < t < \infty$. □

§12.5 **Q16**

Solution.

Let $P = (1, 1, 0), Q = (1, 1, 1)$.
$\overrightarrow{PQ} = (1 - 1)\mathbf{i} + (1 - 1)\mathbf{j} + (1 - 0)\mathbf{k} = \mathbf{k}$
Hence, the line is parallel to $\mathbf{k}$.
A parametrization of the line thought P and Q is
$x = 1, y = 1, z = 0 + t = t, -\infty < t < \infty$.
The line passes through $P(1, 1, 0)$ at $t = 0$ and $Q(1, 1, 1)$ at $t = 1$.
Therefore, a parametrization for the line segment is
$x = 1, y = 1, z = t, 0 \le t \le 1$.



□

§12.5 **Q**22

Solution.

The plane $3x + y + z = 7$ is normal to $\mathbf{n} = 3\mathbf{i} + \mathbf{j} + \mathbf{k}$.
Hence, the plane through $(1, -1, 3)$ parallel to the plane $3x + y + z = 7$ is normal to $\mathbf{n} = 3\mathbf{i} + \mathbf{j} + \mathbf{k}$.
Therefore, the plane is
$3(x - 1) + 1(y - (-1)) + 1(z - 3) = 0$
$3x - 3 + y + 1 + z - 3 = 0$
$3x + y + z = 5$ □

§12.5 **Q**24

Solution.

$(1 - 2, 5 - 4, 7 - 5) = (-1, 1, 2)$
$(-1 - 2, 6 - 4, 8 - 5) = (-3, 2, 3)$
$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 2 \\ -3 & 2 & 3 \end{vmatrix} = (3 - 4)\mathbf{i} - (-3 - (-6))\mathbf{j} + (-2 - (-3))\mathbf{k} = -\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ is normal to the plane.
Since the plane passes through $(2, 4, 5)$ and normal to $-\mathbf{i} - 3\mathbf{j} + \mathbf{k}$, the plane is
$(-1)(x - 2) + (-3)(y - 4) + (1)(z - 5) = 0$
$-x + 2 - 3y + 12 + z - 5 = 0$
$-x - 3y + z + 9 = 0$
$x + 3y - z = 9$ □